Plan of future research

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I would like to continue the study of the topological properties of the space $\mathcal{K}_n$ ($\widetilde{\mathcal{K}}_n$) of (framed) long knots in $\mathbb{R}^n$ ($n > 3$), in particular the double loop space structure and other related features, based on my previous results (see Summary of Research).

(1) In my previous research, using the double loop structure of $\widetilde{\mathcal{K}}_n$, I constructed a (co)homology class of $\widetilde{\mathcal{K}}_n$ in several ways which does not arise from trivalent graphs. I plan to produce more elements of $H^*_{DR}(\mathcal{K}_n)$ from non-trivalent graph cohomology classes via configuration space integrals [1], by generalizing my construction.

(1-1) One of the reason why the research in this direction is interest is the following conjecture due to M. Kontsevich:

Conjecture. When $n > 3$, Vassiliev's spectral sequence converging to $H^*(\mathcal{K}_n)$ [4] degenerates at E^1 over $\mathbb{R}$.

Though the above conjecture has been proved in another way [2], the method of the proof relies on rational homotopy theory. Kontsevich's strategy is as follows; E^1-term bounds $H^*(\mathcal{K}_n, \mathbb{R})$ from above, and the elements which are given by configuration space integrals give a lower bound. Namely it states that all the elements in $H^*(\mathcal{K}_n, \mathbb{R})$ can be obtained by means of iterated integrals, and here the importance of the iterated integrals is shown very straight.

Another importance of this research comes from the fact that the cohomology classes corresponding to trivalent graphs obtained via iterated integrals over configuration spaces are “higher-dimensional analogues” of the perturbative invariants for classical knots and 3-manifolds. In the three-dimensional case, the cohomology classes arising from non-trivalent graphs seem to be less paid attention. But I have already showed that in the case of $\mathcal{K}_n$ ($n > 3$) non-trivalent graphs can produce non-zero cohomology classes. Its analogue in three-dimensional case should be an “invariant” for knots with more precise information. The problems what this “invariant” is, and how many such invariants exist, should be very important. To tackle these problems, I think it is necessary to construct as many elements of $H^*_{DR}(\mathcal{K}_n)$ from non-trivalent graphs as possible.

(1-2) A difficulty of this construction is in the very complicated combinatorics of the graph complex which disturbs efficient computation of the graph cohomology. To overcome this difficulty, I will make use of the geometric information on $\widetilde{\mathcal{K}}_n$, that is, Poisson structures on $H_*(\widetilde{\mathcal{K}}_n)$ induced by actions of operads. Namely, we can expect that similar operations can be defined on the graph cohomology (or on the graph complex) as (the dual of) Poisson algebra structure induced by the operad actions, because it is conjectured that graph cohomology might completely describe $H^*_{DR}(\mathcal{K}_n)$ via the iterated integrals. Applying such operations to the known (co)cycles, we should obtain new (co)cycles. To prove the corresponding (co)cycles of $\widetilde{\mathcal{K}}_n$ are not zero, it suffices to show that the Kronecker product does not vanish, as I have shown in the computation of an example.

(1-3) As one more way to overcome the combinatorial difficulty, I plan to use rational homotopy theory. The graph complex appeared above and Vassiliev's spectral sequence for $H^*(\mathcal{K}_n)$ [4] are analogues of bar complexes which appear in the classical research of loop spaces. In rational homotopy theory many complexes which are equivalent to bar complex have been constructed. I expect that such constructions allow us to reduce our complexes to smaller or computable ones.

(2) As applications of the topology of $\mathcal{K}_n$, I plan to study the following problems. From the homotopy-theoretic viewpoint, $\mathcal{K}_n$ is regarded as a ‘totalization” of all configuration spaces $\mathrm{Conf}\,(\mathbb{R}^n, k)$ [**?**]. The advantage of this viewpoint is that we can describe the spectral sequences for $H^*(\mathcal{K}_n)$ or $\pi_*(\mathcal{K}_n)$ in terms of $\mathrm{Conf}\,(\mathbb{R}^n, k)$'s. Thus we might be able to catch some information of topological objects related to configuration spaces. For example, in case of $\pi_*(\mathcal{K}_n)$, the spectral sequence is written in terms of the Lie algebra consisting of $\pi_*(\mathrm{Conf}\,(\mathbb{R}^n, k))$, that is, the Lie algebra associated with the lower central series of pure braid groups. The collection of the Lie algebras for $k \geq 1$ forms a (co)simplicial group and is an important object in that it relates to Vassiliev invariants for pure braids, the homotopy groups of spheres, and so on. I'd like to reveal how the geometry of $\mathcal{K}_n$ relates to these objects through the spectral sequence.

References

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