Research plan

Geometry, representation theory, and integrable systems arising from Hessenberg varieties

Among several interesting algebraic subsets of the flag variety, there are Springer varieties in geometric representation theory, Peterson varieties in connection with quantum cohomology rings of the flag varieties, and the toric varieties associated with root systems. The *Hessenberg varieties* provides us a unified description of these spaces in the flag variety. In type A_{n-1}, they are defined from an $n \times n$ matrix and a function $h : [n] \to [n]$ satisfying certain properties.

In general, let G be a complex semisimple algebraic group, $B \subset G$ a Borel subgroup, $\mathfrak{g}$ and $\mathfrak{b}$ the Lie algebras of G and B, respectively. For an element $x \in \mathfrak{g}$ and a B-invariant linear subspace $\mathfrak{b} \subseteq H \subseteq \mathfrak{g}$, the Hessenberg variety associated with x and H is defined to be

$$\mathrm{Hess}(x, H) = \{gB \in G/B \mid \mathrm{Ad}(g^{-1})x \in H\}.$$

The following are my research plan.

- **Weyl type character formula for Hessenberg varieties** [In collaboration with Haozhi Zeng and Naoki Fujita]

By restricting a torus equivariant line bundle on the flag variety to a regular semisimple Hessenberg variety, the space of global sections turns out to be a representation of the torus. Its Euler characteristic is described by a formula which naturally generalizes the Weyl character formula. We study the space of global sections on a regular semisimple Hessenberg variety and its higher cohomology groups. This is a first step to the problem whether there is a completely integrable system on regular semisimple Hessenberg varieties.

- **Holomorphic symplectic structure and integrable systems arising from a family of Hessenberg varieties** [In collaboration with Peter Crooks]

There is a special Hessenberg space H_0 given by the direct sum of the Lie algebra of B and the root spaces of the negative simple roots. For example, the Peterson variety and the toric variety associated with the fan consisting of the Weyl chambers are Hessenberg varieties associated with H_0. We denote by $\mathfrak{X}(H_0)$ the family of the Hessenberg varieties associated with H_0. Then it turns out that there is holomorphic Poisson structure on $\mathfrak{X}(H_0)$. The open stratum is symplectomorphic to $G \times S_{\mathrm{reg}}$ admitting a completely integrable system, where S_{reg} is a regular Slodowy slice. We think that the being of the Toda lattice in the background is a key of this phenomenon. We hope to clarify the relationship between $\mathfrak{X}(H_0)$, $G \times S_{\mathrm{reg}}$, and the Toda lattice in terms of holomorphic symplectic geometry.