Research Plan

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1. Differential equations integrable by the sigma function of hyperelliptic curves of genus 3

In [1] dynamical systems on $\mathbb{C}^4$ are introduced on the basis of commuting vector fields on the symmetric square of hyperelliptic curves of genus g. For $g = 3$, we constructed a solution of the systems by the sigma function and two first integrals (publication list 1-1). We will check that the systems are Hamiltonian systems whose Hamiltonians are these first integrals and Liouville integrable. We will delete 3 coordinates of 4 coordinates from the systems and derive partial differential equations, which are integrable by the sigma function. By the change of variables, we can obtain a partial differential equation including unknown functions G_2, G_4 and the two coefficients y_{12}, y_{14} of the defining equation of the curve. If y_{12}, y_{14} tend to 0, then this equation becomes KdV-equation. A solution of G_2 can be constructed by the sigma function. If y_{12}, y_{14} tend to 0, the limit of G_2 is a solution of KdV-equation. We will consider what this function is.

2. Series expansion of the sigma function of arbitrary Riemann surface

The coefficients of series expansion of the Riemann theta function are given by matrices of periods. On the other hand, the coefficients of series expansion of the sigma function around the origin are polynomials of the coefficients of the defining equations of algebraic curves with rational numbers. This fact, which plays an important role in applications, distinguishes the sigma function from the Riemann theta function. This property of the sigma function is proved for telescopic curves and some space curves which are not telescopic (publication list 1-4, [2, 3, 4]). The sigma function is extended to any Riemann surface [5]. A model of algebraic curves, which expresses all algebraic curves, is known, which is called Miura canonical form. Telescopic curves and the space curves considered in [2, 3, 4] are special cases of Miura canonical form. We will express the defining equations of arbitrary algebraic curve by Miura canonical form and show that the coefficients of the series expansion of the sigma funciton of arbitrary Riemann surface [5] are polynomials of the coefficients of the defining equations with rational numbers.

3. Division polynomials of a hyperelliptic Jacobian for any field

For an integer m, we will consider to derive m-torsion points of the algebraic Jacobian (divisor class group) of a hyperelliptic curve of genus g. An element of an affine algebraic Jacobian is expressed by a pair (U, V), a monic polynomial U of degree g and a polynomial V of degree $g-1$ (Mumford representation). We will derive the $2g$ coefficients of U and V such that (U, V) is a m-torsion point. In [6] a necessary and sufficient condition such that (U, V) is a m-torsion point is given as simultaneous equations whose variables are the coefficients of U and V by using the theory of Abelian functions. which is a division polynomial for hyperelliptic curves over $\mathbb{C}$. If the curves are defined over a field of characteristic 0, a division polynomial is obtained by the results of the case $\mathbb{C}$. In [6] it is conjectured that the coefficients of the division polynomials are integers by computer experiments. We will consider the conjecture. If it is proved affirmatively, we will consider whether a division polynomial for a field of positive characteristic is obtained by the reduction of the coefficients. If we obtain the division polynomial for a finite field, we will generalize a method to compute rational numbers of elliptic curves by using division polynomials (ℓ-adic method) to hyperelliptic curves. It is important to assure the safety of hyperelliptic cryptography.

References

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