EXCEPTIONAL SURGERIES ON ALTERNATING KNOTS

市原 一裕　(KAZUHIRO ICHIHARA)

Abstract. The famous Hyperbolic Dehn Surgery Theorem due to W. Thurston says that each hyperbolic knot admits only finitely many Dehn surgeries yielding non-hyperbolic manifolds. Concerning the maximal number of such exceptional surgeries, C. Gordon conjectured that there exist at most 10 for each knot. In this article, we report that this conjecture is true for hyperbolic alternating knots in the 3-sphere.

As a consequence of the famous Geometrization Conjecture raised by W.P. Thurston in [13], all closed orientable 3-manifolds are classified as; reducible (i.e., containing essential 2-spheres), toroidal (i.e., containing essential tori), Seifert fibered (i.e., foliated by circles), or hyperbolic (i.e., admitting a complete Riemannian metric with constant sectional curvature -1). See [11] for a survey.

The recent Perelman's works [7, 8, 9], where he announced an affirmative answer to this Geometrization Conjecture, is now going to become acceptable. Thus, in this article, we will assume that the Geometrization Conjecture can be affirmatively established; this implies that the above classification of closed orientable 3-manifolds can be achieved.

Beyond the classification, one of the next directions in the study of 3-manifolds would be to consider the relationships between 3-manifolds. One of the important operations describing relationships between 3-manifolds is *Dehn surgery* on a knot. That is an operation to create a new 3-manifold from a given one and a given knot (i.e., an embedded simple closed curve) in it as follows: Take an open tubular neighborhood of the knot, remove it, and glue a solid torus back. This gives an interesting subject to study; because, for example, it is known that any pair of closed orientable 3-manifolds are related by a finite sequence of Dehn surgeries on knots. It was proved by Lickorish [6] and Wallace [14] independently.

Another motivation to study Dehn surgery comes from the following famous fact, now called the Hyperbolic Dehn Surgery Theorem, due to W.P. Thurston [12]: On a hyperbolic knot (i.e., a knot with hyperbolic complement), all but finitely many Dehn surgeries yield hyperbolic 3-manifolds. In view of this, such finitely many exceptions are called *exceptional surgeries*. Then It is natural to ask:

Question. *How many exceptional surgeries can occur on a knot?*

Concerning this question, C.McA. Gordon conjectured that there exist at most 10 exceptional surgeries on each hyperbolic knot. See [4, Problem 1.77]. As far as the author knows, the sharpest known bound is; they are at most 12, which is obtained as a corollary of the so-called "6-theorem" given by Agol [1] and Lackenby [5] independently. We remark that, if we does not assume the Geometrization "Theorem", then the best known is; at most 60, given by Hodgson and Kerckhoff [2].

Then our main result is the following.

Main theorem. *On a hyperbolic alternating knot in the* 3*-sphere, there exist at most* 10 *exceptional surgeries.*

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Therefore the Gordon's conjecture is true for such knots. Here a knot is called *alternating* if it admits a diagram with alternatively arranged over-crossings and under-crossings running along it.

Note that Gordon also conjectured that a hyperbolic knot with 10 exceptional surgeries must be the well-known figure-eight knot in the 3-sphere S^3 only. The figure-eight knot is also alternating, but our argument cannot tell that it is the only knot admitting 10 exceptional surgeries.

The theorem above follows from the next;

Theorem 1. *On a hyperbolic alternating knot in the* 3*-sphere, all non-trivial exceptional surgeries are integral.*

together with the author's previous work;

Theorem 2 (c.f., [3])**.** *On any hyperbolic knot in the* 3*-sphere, there are at most* 9 *integral exceptional surgeries.*

Here we recall fundamental terminologies. See [10] in details for example. As usual, by a *slope*, we call an isotopy class of a non-trivial unoriented simple closed curve on a torus. Then Dehn surgery on a knot K is characterized by the slope on the peripheral torus of K which is represented by the simple closed curve identified with the meridian of the attached solid torus via the surgery. When K is a knot in S^3, by using the standard meridian-longitude system, slopes on the peripheral torus are parametrized by rational numbers with $1/0$. For example, the meridian of K corresponds to $1/0$ and the longitude to 0. By the *trivial* Dehn surgery on K in S^3, we mean the Dehn surgery on K along the meridional slope $1/0$. Thus it yields S^3 again, which is obviously exceptional. We say that a Dehn surgery on K in S^3 is *integral* if it is along an integral slope. This means that the curve representing the slope runs longitudinally once.

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630–8528 奈良市高畑町　奈良教育大学教育学部数学教育講座. Department of Mathematics Education, Nara University of Education, Takabatake-cho, Nara, 630–8528.

E-mail address: ichihara@nara-edu.ac.jp