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1 Brauer

Richard Brauer initiated the concept of "local to global" information, i.e. the idea that information about a p-block of a finite group G would be encoded in (known) information about a corresponding block (or blocks) of a local subgroup, i.e. the normalizer of a p-subgroup. In particular one would look at the normalizer of the defect group of the block.

In 1942 Brauer published a paper where he took $|G| = pm$ where $(p, m) = 1$. Then a Sylow p-subgroup P has order p. Let $N = N_G(P)$. The p-blocks of G have defect group either 1 or P. Arrange the character table Z of G as

Z_1	Z_2
Z_3	0

where the first (resp. second) block of rows consists of characters of degree prime to p (resp. divisible by p) and the first (resp. second) block of columns consists of p-regular classes (resp. p-singular classes. Also arrange the character table Z^* of N in a similar way.

Z_1^*	Z_2^*

Brauer proved:

(i) The number of characters of G of degree prime to p is equal to the number of characters of N,

(ii) The matrix Z_2 is the same as Z_2^* except that some rows are to be multiplied by -1.

This theorem has led to the modern concept of "isotypy" due to Brouè. We give below some examples of this phenomenon.

We have the following table for $GL(3,2)$.

order	1	2	3	4	7	7
size	1	21	56	42	24	24
χ_1	1	1	1	1	1	1
χ_2	6	2	0	0	-1	-1
χ_3	7	-1	1	-1	0	0
χ_4	8	0	-1	0	1	1
χ_5	3	-1	0	1	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$
χ_6	3	-1	0	1	$\frac{-1-i\sqrt{7}}{2}$	$\frac{-1+i\sqrt{7}}{2}$

We next look at the character tables for $N(P_3)$ and $N(P_7)$ where P_3 and P_7 are Sylow 3- and 7- subgroups.

order of element	1	2	3
classsize	1	3	2
ψ_1	1	1	1
ψ_2	1	-1	1
ψ_3	2	0	-1

order of element	1	3	3	7	7
classsize	1	7	7	3	3
ψ_1	1	1	1	1	1
ψ_2	1	ζ	ζ^2	1	1
ψ_3	1	ζ^2	ζ	1	1
ψ_4	3	0	0	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$
ψ_5	3	0	0	$\frac{-1-i\sqrt{7}}{2}$	$\frac{-1+i\sqrt{7}}{2}$

2 Harish-Chandra

References: Curtis and Reiner, Methods of Representation Theory, I and II
Carter, Finite Groups of Lie type

Inspired by the representation theory of real Lie groups, Harish-Chandra introduced a way of classifying the irreducible characters of a finite reductive group into families.

Here G is a connected, reductive algebraic group defined over F_q, $F : G \to G$ is a Frobenius morphism and G^F the finite reductive group of F-fixed points of G. Let $L \leq P \leq G$, where P is a F-stable parabolic subgroup of G and L is an F-fixed Levi subgroup of P. Let $\mathcal{C}(G^F)$ be the space of complex-valued class functions on G. Then Harish-Chandra induction is a map

$$R_L^G : \mathcal{C}(L^F) \to \mathcal{C}(G^F)$$

defined as follows. If $\psi \in \mathrm{Irr}(L^F)$ then $R_L^G(\psi) = \mathrm{Ind}_{P^F}^{G^F}(\psi\ )$ where $\psi\ $ is the character of P^F obtained by inflating ψ to P^F.

Definition. $\chi \in \mathrm{Irr}(G^F)$ is cuspidal if $\langle \chi, R_L^G(\psi) \rangle = 0$ for any $L \leq P$ where P is a proper parabolic subgroup of G. We call $(L, \theta$ a cuspidal pair if $\theta \in \mathrm{Irr}(L^F)$ is cuspidal. The main theorem here is as follows.

THEOREM. (i) Let (L, θ), (L', θ') be cuspidal pairs. Then $\langle R_L^G(\theta), R_{L'}^G(\theta') \rangle = 0$ unless the pairs (L, θ), (L', θ') are G^F-conjugate.

(ii) If $\chi \in \mathrm{Irr}(G^F)$, then $\langle \chi, R_L^G(\theta) \rangle \neq 0$ for a cuspidal pair (L, θ) which is unique up to G^F-conjugacy.

Thus $\mathrm{Irr}(G^F)$ is partitioned into Harish-Chandra families.

The endomorphism algebras of the modules for $R_L^G(\theta)$ where (L, θ) is a cuspidal pair have been described by Curtis-Iwahori-Kilmoyer, Howlett-Lehrer, Lusztig and Geck.

As a result we get the following. Let (L, θ) be a cuspidal pair. Then there is a bijection

$$\{\text{Constituents of } R_L^G(\theta)\} \longleftrightarrow \{\text{Characters of } W_{G_F}(L, \theta) = N_{G^F}(L, \theta)/L^F\}$$

.

The group $W_{G_F}(L, \theta)$ is "almost" a Coxeter group. When P is a Borel subgroup and $\theta = 1$ it is equal to W^F. We call $W_{G_F}(L, \theta)$ a relative Weyl group.

3 Deligne-Lusztig

References: Carter, Digne-Michel, Representations of finite groups of Lie type, LMSST 21

We continue with G^F a finite reductive group. The idea is to generalize Harish-Chandra induction to the case where we have $L \leq P \leq G$, where L is an F-fixed Levi subgroup of the parabolic subgroup P but P is not necessarily F-stable. For example, L could be an F-stable maximal torus. Let l be a prime not dividing q. We now define $\mathcal{C}(G^F)$ to be the space of $\bar{Q}_l$-valued class functions on G. Then Lusztig (generalizing the original map of Deligne-Lusztig) defined a map

$$R_L^G : \mathcal{C}(L^F) \to \mathcal{C}(G^F)$$

which takes generalized characters to generalized characters, using l-adic cohomology.

Definition. $\chi \in \mathrm{Irr}(G^F)$ is unipotent if $\langle \chi, R_T^G(\theta) \rangle \neq 0$ for some F-stable maximal torus T of G and a character θ of T^F.

In his book Lusztig gave a classification of the characters of G^F when G has connected center. In particular the classification of unipotent characters is done for arbitrary finite reductive groups, and has the remarkable property that it depends only on certain "root data" associated with the group and not on q. This led Broué, Malle and Michel to define the concept of a "generic group".

Example: GL_n : The unipotent characters, which in this case are all constituents of $\mathrm{Ind}_{B_F}^{G_F}(1)$ where B is a Borel subgroup, are parametrized by partitions of n.

The irreducible characters of G^F are partitioned into "geometric conjugacy classes" parametrized by F-stable semi-simple confugacy classes of a dual group G^*. If (t) is such a class, we denote the corresponding geometric conjugacy class by $\mathcal{E}(G^F, (t)$.

$\chi \in \mathrm{Irr}(G^F)$ belongs to $\mathcal{E}(G^F, (t)$ and $C_G^*(t)^0$ is a Levi subgroup of G^* then there is a subgroup $G(t)$ of G in duality with $C_G^*(t)^0$ and a unipotent character ψ of $G(t)^F$ such that $\langle \chi, R_{G(t)}^G(\psi) \rangle \neq 0$. We say that χ corresponds to the pair $(t), \psi$ under Jordan decomposition.

4 e-Harish-Chandra

Reference:[BMM] Broué, Malle, Michel; [CE] Cabanes-Enguehard, Representation Theory of finite reductive groups, Cambridge University Press, 2004.

Let $\phi_e(q)$ denote the e-th cyclotomic polynomial. The order of G^F is a polynomial in q, the product of a power of q and certain cyclotomic polynomials. A torus T of G is called a ϕ_e-torus if T^F has order a power of $\phi_e(q)$.

Example. In GL_n, it is convenient to think of T with $|T^F|$ a power of $q^e - 1$ as a ϕ_e-torus.

Then the centralizer in G of a ϕ_e-torus is called an e-split Levi subgroup of G.

Example. In GL_n, if L is e-split Levi subgroup then L^F is isomorphic to a group of the form $\prod_i GL(m_i, q^e) \times GL(r, q)$.

An e-cuspidal pair (L, θ) is defined as in the Harish-Chandra case, using only e-split Levi subgroups. Thus $\chi \in \mathrm{Irr}(G^F)$ is e-cuspidal if $\langle \chi, R_L^G(\psi) \rangle = 0$ for any e-split Levi subgroup L.

Let L be e-split, (L, θ) a unipotent e-cuspidal pair. Using Lusztig's classification of unipotent characters and the explicit description of the R_L^G map by Asai, Shoji, Schewe and others, [BMM] show the following:

The unipotent characters of G^F are divided into e-Harish-Chandra families, as in the usual Harish-Chandra case of $e = 1$. There is a bijection

$$\{\text{Constituents of } R_L^G(\theta)\} \longleftrightarrow \{\text{Characters of } W_{G^F}(L, \theta) = N_{G^F}(L, \theta)/L^F\}$$

.

In this case, $W_{G_F}(L, \theta)$ is a complex reflection group.

Example. In GL_n, let χ_λ be a unipotent character corresponding to the partition λ of n occurring in $R_L^G(1)$. Then $W_{G_F}(L)$ is isomorphic to the wreath product of Z_e with S_r for some r, and the bijection is the classical $\lambda \to e-$quotient of λ.

5 Unipotent Blocks

As before l is a prime not dividing q. Let e be the order of q mod l. A unipotent l-block of G^F is a block (in the sense of Brauer) which contains unipotent characters. The work of Fong-Srinivasan, [BMM] and Cabanes-Enguehard together shows:

Let B be a unipotent l-block. There exists a unipotent e-cuspidal pair (L, λ), unique up to G^F-conjugacy, such that there is a bijection

$$\{\text{Unipotent Characters in B}\} \longleftrightarrow \{\text{Constituents of } R_L^G(\lambda)\} \longleftrightarrow \{\text{Characters of } W_{G_F}(L, \lambda)\}$$

.

Our work starts with trying to find a similar result for the other, non-unipotent characters in B. The Dade conjecture will follow from such a "local-to-global" bijection, modified to include blocks of subgroups of G^F of the form $N_{G^F}(M)$ where M is an e-split Levi subgroup.

Definition. $B_{(t)}$ is the intersection of B with a geometric conjugacy class $\mathcal{E}(G^F, (t)$, where (t) is a F-stable conjugacy class of l-elements of G^*.

Note that B_1 has been described above. We wish to describe $B_{(t)}$ for all t. In the case when the defect group of B is abelian this has been done by [BMM]' as follows:

If (L, λ) is as above, every character in B is a constituent of some $R_L^G(\theta\lambda)$ where θ varies over certain linear characters of L^F of l-power order.

Hence, in the abelian defect group case, in some sense the group $Z^0(L)^F W_{G_F}(L, \lambda)$ gives the information on the block.

This is not true in the non-abelian case. However, by a result of Cabanes-Enguehard we know that the characters in $B_{(t)}$ are constituents of twisted-induced characters of the form $R_{G(t)}^G(\hat{t}\lambda_t)$ where $G(t)$ is a Levi subgroup (not necessarily e-split) of G in duality with $C_G^*(t)^0$, $\hat{t}$ is a linear character of $G(t)^F$ in duality with t. Furthermore ψ is a unipotent character of $G(t)^F$ in a block which corresponds to an e-cuspidal pair (L_t, λ_t) of $G(t)^F$. The constituents of $R_{G(t)}^G(\hat{t}\lambda_t)$ are then constituents of $R_{L_t}^G(\hat{t}\lambda_t)$, which bears a similarity to the abelian defect group case.

We define K_t to be the smallest e-split Levi subgroup of G containing L_t.

Assumptions: (i) l is odd, good, and $l \neq 3$ if ${}^3D_4$ occurs in G,

(ii) $C_{G^*}(t)$ is connected,

(iii)The R_L^G map and Jordan decomposition of characters commute.

With these assumptions we can show:

(i) Let $\theta_t = R_{L_t}^{K_t}(\hat{t}\lambda_t)$. Then (K_t, θ_t) is an e-cuspidal pair of G.

(ii) The characters in $\mathcal{E}(G^F, (t))$ are divided into e-Harish-Chandra families.

(iii) There is a bijection

$$\{\text{Characters in } \mathrm{B}_{(\mathrm{t})}\} \longleftrightarrow \{\text{Constituents of } R_{K_t}^G(\theta_t)\} \longleftrightarrow \{\text{Characters of } W_{G_F}(K_t, \theta_t)\}$$

.

Remark. These assumptions appear in [CE, Theorems 22.9, 23.2] and in Cabanes-Enguehard, Local methods for blocks of reductive groups over a finite field, in Proc. Conference in Luminy, Progress in Mathematics vol. 141, Birkhauser, 1997, Chapter 2.

Thus, for instance in the non-abelian defect group case, several subgroups K_t have to be considered to get the global information. It is hoped that this will give a clue to modifying Broué's abelian defect group conjecture to the general case.