Research plan (April 2024 ∼)

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The study of elliptic operators in the framework of global setting brought deep understandings of the manifold theory and analysis. In my coming research activity, I am going to study global phenomena of mainly sub-elliptic operators including elliptic cases.

Laplace-Beltrami operator and Dirac operator are defined by means of geometric structures on manifolds, the first one is a second order elliptic differential operator and the second one is a first order elliptic differential operator, respectively. In the study here sub-Riemannian structures and associated sub-Laplacians are the main subjects. A sub-Laplacian is a sub-elliptic second order differential operator and is defined based on a sub-Riemannian structure.

This geometric structure (= sub-Riemannian structure) requires that there exists a bracket generating sub-bundle in the tangent bundle.

The opposite structure, that is the foliation, was studied since many years ago. On the other hand until recently it has not been so much studied of such structure from geometric and global analytic point of view, so that I am expecting that the research of this subject has an enough meaning even by comparing with Riemannian manifolds theory and analysis on them.

Contact manifolds and nilpotent Lie groups are basic examples of manifolds carrying a sub-Riemannian structure and it happens often that the total space of a Riemannian submersion has both structures, foliation and sub-Riemannian. Hence there are ample examples of such manifolds to be studied.

Although we can define a transversely elliptic operator on foliated manifolds, there are no natural differential operator associated to the foliation structure. On the other hand there is an intrinsically defined second order differential operator (we call it a sub-Laplacian) on sub-Riemannian manifolds reflecting the sub-Riemannian structure. Hence, again there must be enough meaning to study the sub-Laplacian in contrast with the Laplacian in the Riemannian case.

Since this operator satisfies the "*sub-elliptic estimate*" (= sub-ellipticity), which was proved by Hörmander and the basic structure of the spectrum is similar to elliptic operator cases. However, there is non trivial characteristic variety so that from the geometric point of view it is not enough to treat them in the topological framework(K-theory). This fact may include difficulty in the study, and at the same time we can expect the possibility of new phenomena other than obtained by the property of ellipticity. So, I am going to continue the research on these topics under the direction to find possible new phenomena which will not be included in the elliptic operator theory, and together including the problems whether named classical manifolds have this structure and analytically similar properties of this type operator with the elliptic cases, like Weyl low or an explicit construction of the heat kernel.

Although there were strong restriction in the research activity by the covid-19 pandemic in the passed several years, now I expect I can continue joint research work with my former coworkers and am planing the research on the following explicit problems (a) $\sim$ (d) **:**

(a) Nilpotent Lie groups are examples of sub-Riemannian manifolds carrying a "good" sub-Riemannian structure, which is called equi-regular sub-Riemannian structure. Among them I was studying a class of algebras (and groups) attached to Clifford algebras, which we call pseudo H-type algebras and groups in these years. As a next step, I am going to start to construct and classify "*orthonormal invariant lattices*". Such lattices (=uniform discrete subgroups) are a special type among integral lattices, even so their classification are still unclear.

In relation with these groups, I will study sub-Laplacians on their compact nil-manifolds, their spectral zeta functions and zeta regularized determinants.

In particular, I am going to study the inverse spectral problem of the sub-Laplacians from a point of a classical famous problem relating with the Riemann zeta function which must be interesting, since the residues of the spectral zeta function of the sub-Laplacian of pseudo H-type nil-manifolds relate with the values of Riemann zeta function at integral points in many cases.

(b) At the same time I started a study of Radon transformation from the point of the Fourier integral operator theory. It turned out that this problem highly related with the problem (d) below.

(c) Since the autumn 2020, I had started a problem on the construction of the Green kernel of a sub-Laplacian on a manifold with conic singularity with a colaborator by the method of symbolic calculus. After the restriction by the corona-pandemic I am expecting to be possible to make some progress.

(d) Although Lie groups have always invariant sub-Riemannian structures, it is not clear whether their symmetric spaces have such a structure always or not. So if I have a time I will try to construct explicitly such a structure or classify them for some cases or a particular symmetric space.