Research Results

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Transition process of factor graphs of Sturmian words

The **Sturmian words (sequences)**, defined as aperiodic infinite words of minimal complexity, share many interesting properties and can be defined differently.

In this study, we elucidated the properties of Sturmian words using the **factor graph** introduced by Rauzy (1982) [2]. We focus on the transition process of the factor graph for the length of factors and examine in detail the case where the shape of the graph changes when the order of the factor graph increases by one. We have improved our perspective by using the Farey sequence and considering the entire real number set, including rational numbers as well as irrational numbers. Furthermore, a comprehensive understanding of mechanical words was obtained by examining the properties of those using geometric mappings introduced by Yasutomi (1998)[3] and Akiyama (2021)[1].

In this study, we focused on the fact that the division of the central path, the 0-side path, and the 1-side path in the factor graph changes continuously when α is fixed and colored the three divisions correspondingly.

If α coincides with any fraction $\frac{p}{q}$ in the Farey sequence $Farey_{n+1}$, then the factor graph is a loop of q vertices and the line segment $[0,1) \times \{1-\alpha\}$ is equally divided into q. Otherwise, the factor graph is determined by the Farey pair surrounding α in $Farey_{n+1}$.

We can see how the shape of the factor graph of order n changes with the value of α when each convex cell obtained by this partition is colored to distinguish the central path, the 0-side path, and the 1-side path, respectively.

The main results are as follows.

- The symmetry of factor graphs.
- The factor graph of a mechanical word is determined by its slope α, the order n of the factor graph, and the Farey pairs surrounding α in the Farey sequence $Farey_{n+1}$.
- Given three nonnegative integers $k \geq 1,\ l \geq 0,\ m \geq 0\ (l+m>0)$, a necessary and sufficient condition for the existence of a factor graph with k, l and m vertices on the central path, 0-side path and 1-side path, respectively.
- Relationship between the slope α and the intercept ρ, and the position of the prefix of the mechanical word of length n in the factor graph determined by α and ρ.
- When the mechanical word $S_{\alpha,\rho}$ has a bispecial factor of length n, the sum of possible lengths of values of $\alpha \in [0,1]$ and its asymptotic formula, the sum of the areas of the possible values of $(\rho,\alpha) \in [0,1) \times [0,1]$ and its asymptotic formula, as well as the area of each of the three paths of the factor graph in the geometric mapping and its asymptotic formula.

We have also found new proof s of the three distance theorem and its dual, the three gap theorem, which plays an essential role s in Sturmian words and Diophantine approximation, and have posted it on arXiv. In addition, we are in the process of writing a paper for submission in parallel with a paper on the transition process of factor graphs.

References

[1] S. Akiyama, *Asymptotic formula for balanced words*, Journal of Number Theory, (2021).

[2] G. Rauzy. *Suites à termes dans un alphabet fini*, Séminaire de Théorie des Nombres de Bordeaux (1982): 1-16.

[3] S. Yasutomi. *The continued fraction expansion of α with $\mu(\alpha)=3$*, Acta Arith. **84** (1998): no. 4, 337-374.