3 Application to automorphic representations

The basic motivation of Paper 2 and Preprints 3 and 4 is to give integral forms of automorphic representations and their cohomology. Let me explain its background.

The cohomologically induced modules are characterized as irreducible essentially unitary $(\mathfrak{g}, K)$-modules with nontrivial cohomology (Vogan–Zuckerman). As its consequence in automorphic representation theory, we see that **cohomologically induced modules appear as the infinite part of cohomological cuspidal automorphic representations**. With the finite part, special values of automorphic L-functions appear in their cohomology. For instance, let us think of the classical one variable holomorphic modular forms. Then we obtain the so-called **period** which is a complex number by comparing the natural rational structures of the cohomology and the rational structure obtained from the special L-values. Using this, one can **split special L-values into the period, certain powers of π, and algebraic numbers**. This is called the rationality of special L-values. If we think of the discriminant function Δ, we have

$$\frac{L(n,\Delta)}{(2\pi\sqrt{-1})^n\Omega^{(\pm 1)^n}(\Delta)} \in \mathbb{Q}^\times \quad (1 \leq n \leq 12).$$

Here $L(s, \Delta)$ is the automorphic L-function attached to Δ, and $\Omega^{\pm 1}(\Delta)$ are periods. Towards rationality of special values of other automorphic L-functions, Raghuram–Shahidi proposed a formulation of the period in terms of the rational structure of the cohomology of automorphic representations of GL_n. Recently, Harder–Raghuram have proposed its integral analog.

4 Study on orbit decomposition

I wish to get **smaller arithmetic forms of K-orbit decomposition** by proving that the quotient of the flag scheme by K (generalization of Paper 1 and Preprint 4). However, **this is not true because of the nontrivial closure relations among orbits**. In SGA 3, the representability problem of partial flag schemes by parabolic subgroups was resoloved by introducing the moduli scheme of parabolic subgroups with standard position. In fact, this moduli scheme is étale locally the disjoint union of orbits as a scheme. I prove that **the quotient of the moduli space of Borel subgroups B with the property that $(B, \theta(B))$ are of standard position by K is representable**, where θ is the involution corresponding to K. For this, I prove that the closure relation in this moduli space is étale locally trivial. Then I lift Matuski and Richardson–Springer's combinatorial description of the K-orbit decomposition over algebraically closed fields of characteristic not two to **give an orbit decomposition étale locally**. As a result, the representability of the quotient by K follows. I also prove that **models of orbits are imbedded affinely into the flag scheme** by rephrasing Beilinson–Bernstein's proof on the fact that the K-orbits over $\mathbb{C}$ are imbedded affinely into the flag variety in terms of this moduli space.