Future reserch

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Recently, Applebaum and Mohari have shown that the Lévy process on compact Riemannian manifolds is symmetric Markov process under suitable conditions. Moreover, the Lévy process on non-compact Riemannian manifolds is strongly expected to be Markovian. Considering these situations, I'd like to study the following problems:

Problem 1 Obtain the heat kernel estimates for the Lévy process on Riemannian manifolds.

Problem 2 Study the relation between the path-properties of the Lévy process on Riemannian manifolds and their curvature.

Problem 3 Find the invariant measure for the Lévy process and show the large deviation principle. Moreover, if the Lévy measure is transient, try to obtain the escape rate.

From now on, I'd like to explain the significance to solve the problems and how to tackle them.

Problem 1: Solving this problem implies that we can understand how the curvature affects the Lévy process on Riemannian manifolds. I shall investigate the conditions under which the Lévy process on non-compact Riemann manifolds is a symmetric Markov process, and analyze the properties of such process using the Dirichlet forms. The Dirichlet forms has been used by Chen-Kim-Kumagai (2009) and others to evaluate the heat kernel of a symmetric jump process, and I expect that the problem 1 can be solved by applying these results.

Problem 2: In my previous work (Kai 2024), I showed that the Lévy process on Cartan-Hadamard manifolds is transient and conservative. On the other hand, the question of what kind of metric on a Riemann manifold makes the Lévy process recurrent is still open. To complement our previous work, I set this research question. From Problem 1, the two sided heat kernel estimates will be obtained. Recurrence and transience can be distinguished by whether $\int_0^\infty p(t,x,y)dy$ is convergent or divergent.

Problem 3: Once Problem 2 is solved, we know the conditions for a manifold whose Lévy process has an invariant measure. To find the Donsker-Varadhan type large deviation principle for the Lévy process $\{X_t;\ 0 \leq t < \infty\}$ tells us how fast the empirical distribution diverges to the invariant measure. The large deviation principle for the symmetric Lévy processes on Euclid space was studied in Takeda-Tsuchida (2011) by using the Dirichlet forms. If I can generalize this result and extend it to the Lévy processes on Riemannian manifolds, I will be able to solve this research problem. If the Lévy process is transient, the escape rate of the process toward the infinity point can be obtained by evaluating the radial part of the process. In fact, the escape rate of the Lévy process on a Cartan-Hadamard manifold whose sectional curvature is pinched by negative constants was found by Kai (2024). I aim to further generalize this result by solving this research problem.