Research project

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There are two types of hierachical structures of algebraic geometry, which I would like to investigate:

(A) Suppose the base field k is perfect, and consider the following naive **hierarchical structure** which interpolates the rationality and the ruledness (here $\overset{\text{bir}}{\sim}$ stands for the birational equivalence):

$$\begin{cases} X \text{ is } \underline{(-i)\text{-rational}} \text{ or } \underline{(n-i)\text{-ruled}} & \text{for } 0 \leq i \leq n-1, \\ \text{if } \exists \text{ an } i\text{-dimensional variety } Z^i \text{ s.t. } \mathbb{P}^{n-i} \times Z^i \overset{\text{bir}}{\sim} X. \end{cases} \tag{1}$$

Then, observe that this **hierachical structure** allows us to upgrade the well-strudied hierarchical structure interpolating the rationality and the rationally ever since Lüroth:

$$\begin{aligned} (-i)\text{-rational} &\implies \text{stable } (-i)\text{-rational} \implies \text{ratract } (-i)\text{-rational} \\ &\implies \text{separably } (-i)\text{-unirational} \implies \text{separably } (-i)\text{-rationally connected} \\ &\implies (-i)\text{-rationally connected} \end{aligned}$$

and investigate it.

(B) Introduce the following **hierarchical structure** which interpolates the birational geometry and the biregular geometry:

Let us call two equi-dimensional X, Y be codimension $> c$ birational equivalent, if there are dense opens $U \subset X, \quad V \subset Y$ such that

$$\begin{cases} \operatorname{codim}_X (X \setminus U) > c, \qquad \operatorname{codim}_Y (Y \setminus Y) > c \\ X \supset U \cong V \subset Y \end{cases}$$

Of course, when $c = 0$, this equivalence relation is nothing but the usual birational equivalence, and approaches to the biregular equivalence as c increases. I am interested in studying invariants of this hierarchical equivalence relation and related equivalence relation.

Concerning the hierchical structure of the type **(A)**, I have already obtained several results and announced them in some mathematical meetings. Amongst of all, I uploaded the following paper to arxiv:

- Norihiko Minami, Generalized Lüroth problems, hierarchized I: SBNR
- stably birationalized unramified sheaves and lower retract rationality, **arXiv:2210.12225**

Here, for any unramied Zariski sheaf S, which Morel introduced in his masterpiece: "$\mathbb{A}^1$-Algebraic Topology over a Field, Springer Lecture Notes in Mathematics 2012,"I constructed s birational subsheaf $S_{sb} (\subseteq S)$, which is shown to be stably birational invariant and $S_{sb}(X) = S(X)$ for any smooth proper scheme X, both of which follow from (different) remarks of Colliot-Thélène.

Furthermore, thanks to Shuji Saito and Junnosuke Koizumi, Morel's unramified sheaf is known to include the very general recipricity sheaves defined by Kahn-Saito-Yamazaki(ComposMath 2016).

For the proof of the constructing subsheaf $S_{sb}(\subset S)$, the method which Novacoski-Spivakovsky developed in order to prove their local uniformization theorem (Josnei Novacoski, Mark Spivakovsky, Reduction of local uniformization to the rank one case, Valuation theory in interaction, 404–431, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2014) plays an essential role.

On the other hand, as an example of the hierarchical structure of type **(B)**, I have been investigating to interprete the torsion obstruction for the integral Hodge/Tate conjecture as hierarchical invariants interpolating the birational invariance and the biregular invariance corresponding to the codimensions, and obtained some fairly general result. So, I am now writing on this result.

By the way, if we apply some simple concept introduced in this work, we realize the importance of those schemes which enjoy the integral Hodge/Tate conjectures. It is well-known that those schemes. which admit cellular decomposition like the projective space, smooth projective toric varieties, and flag varieties, are such examples. On the other hand, Voisin's integral Hodge conjecture theorem (ASPM 2006) for 3 dimensional uniruled varieties (which are nothing but the Kodaira dimension $-\infty$ 3-folds) and Totaro's integral Hodge conjecture theorem (JIMJusseiu 2021) for some Kodaira dimension 0 3-folds under some restriction (without which some counter-examples are known), both of which are shown by applying the minimal model program theory for 3-folds, appear to be very deep and expected to present interesting applications even for general dimensions.

Here, we find a very interesting application of minimal model program theory for 3-folds, which is appori expected to have its applications only in the setting of birational geometry and 3 folds, now to invariants to interpolate the birational geometry and the biregular geometry, in the general dimensions.

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