Summary of research results to date

I majored algebraic geometry in my master program, and became interested in the theme of the moduli of quiver representations and their wall-crossing phenomena.

Around that time, I heard a lecture by Kota Yoshioka at the Mathematical Society of Japan, and learned about Nakajima-Yoshioka's research on investigating the moduli of vector bundles on algebraic surfaces, particularly the projective plane $\mathbf{P}^2$, using wall-crossing phenomena. Their research did not directly investigate the wall-crossing phenomenon on the projective plane $\mathbf{P}^2$, but investigated it through its one-point blow-up.

Two subjects that the applicant is interested in, quiver expressions and vector bundles, were linked by the common keyword of wall-crossing phenomenon. Then, when I was leaving the venue, I came up with the idea of using the quiver representations to directly investigate the wall-crossing phenomenon on the projective plane.

From a certain point of view in terms of the root systems, Nakajima-Yoshioka investigated a wall that is orthogonal to real roots. On the other hand, the applicant focused on the wall that is orthogonal to the imaginary root. When dealing with infinitely connected components of a moduli, there are countless roots to consider, but since all imaginary roots are proportional, there is only one orthogonal wall. I investigated in detail how the moduli change across over this one wall. My curiosity about this phenomenon was the main motivation for my research.

Paper list 4 and 5:

As a result of papers 1 and 3, I received a manuscript from Prof. Takuro Mochizuki for his book summarizing the theory of the analysis method for wall-crossing phenomena developed by him. When I was exploring the application of this idea and reporting to my host Prof. Hiraku Nakajima, I learned about the Ito-Marukichi-Okuda conjecture in the field of mathematical physics. By slightly changing their settings, we found a functional equation that describes the symmetry of the Nekrasov partition function. First, in the paper 4, we provided a proof for this functional equation. Furthermore, in the paper 5, we proved the Ito-Marukichi-Okuda conjecture. This conjecture is a functional equality between the orbifold version of the Nekrasov function and the zero-order Hirota differential. For higher-order Hirota differentials, geometrically it is necessary to take the slant product, and there is still room for research. Both papers use the wall-crossing formula for the proof.

Paper list 8:

We investigated previous research on the quantum q difference Painlevé equation of the Type VI and the relationship with the Shakirov equation. In particular, the $D^{(1)}_5$ Weyl group symmetry of the Becklund transformation of the Type VI Painlevé equation had already been quantized by co-author Prof. Hasegawa. In this paper, we show that the realization of this quantization operator on the function space is given by the Shakirov equation. Furthermore, we expect that the solution to the Shakirov equation is given by the Affin-Laumon partition function. The combination of Shakirov equations derived through this prediction process is compatible with the wall-crossing formul, and we expect future progress.

Paper list 9 and 12

So far I derived the wall-crossing formula from scratch for each partition function. Therefore, in the paper 9, we summarized the wall-crossing formula in the general setting of framed quiver representation. Furthermore, in the paper

12, we derived the K-theoretic wall-crossing formula. As an application of this, we derived the functional equation of the K-theory version of the A_1-type Laumon partition function and provided another proof of the Kajihara transformation formula. What is distinctive about K-theory is not the partition function itself, but the fact that the q-Borel transformation of the partition function becomes a multiple hypergeometric series that satisfies the transformation formula. From this, I hope that it will be possible to derive the q-difference equation using the wall-crossing formula. I am thinking about affinization to get a perspective for that. At present, we have predicted the functional equation that the Affine Laumon partition function should satisfy through joint research with Prof. Shiraishi.