FUTURE RESEARCH PLANS

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The problem I would like to address in future research is to clarify the relationship between the geometric structures of manifolds in dimensions four and higher and knots. I plan to tackle the following three major tasks for that.

(1) Complete classification of $\tau_{m,n}(K)$ for the cases where$K$ is torus knot and $m = 2$.
(2) Classify $k + 1$ knots obtained from 1-knots by twist sppining k times.
(3) Construct higher-dimensional branched twist spins.

For (1) and (2),as the applicant mentioned in "summary on my reseach", he has obtained results about classification of branched twist spins and has also obtained about the clasification on 3-knots. Let $\tau_{m_k}(\tau_{mk-1}(\dots\tau_{m_1}(K)\dots)) = \tau_{\mathbf{m}}(K)$ be the $k + 1$-knot obtained from a 1-knot K by twist sppning k times. Its knot group $G(\tau_{\mathbf{m}}(K))$ has the presentation

$$G(\tau_{\mathbf{m}}(K)) \cong \left\langle x_1, \dots, x_l, h_j \;\middle|\; \begin{array}{l} r_1, \dots, r_l, \\ x_i h_j x_i^{-1} h_j^{-1}, \\ x_1^{m_j} h_j, \end{array} \right\rangle \quad (j = 1, \dots, k).$$

If K_1 and K_2 are non-equivalent non-torus knots, $m = \gcd(m_1, \dots, m_k)$, and $Z(\pi_1^{orb}\mathcal{O}(K_i, m))$, then $\tau_{\mathbf{m}}(K_1)$ and $\tau_{\mathbf{m}}(K_2)$ are not equivalent. However, as k increases, this result is further from a complete classification. Thus he will first study on the case where K_1 and K_2 are torus knots and $m = 2$ and will obtain precise classification about branched twist spins. As a remark, The assumptions in Theorem 3 is used for isomorphism between the quotient of the knot group by its center and the orbifold fundamental group and it is not known whether it is essential or not for general case. More specifically, $Z(\pi_1^{orb}(\mathcal{O}(K, m)))$ is trivial for the case where $m \geq 3$ and K is hyperbolic or satellite however it is not know wether $Z(\pi_1^{orb}(\mathcal{O}(K, m)))$ is trivial or not for a toru knot or a composite knot. After classifying such cases, we can study about (2) as an analogy of the observation above.

For (3), There are two definitions of branched twist spins: one is the preimage of $\tau_m(K)$ by the n-th fold branched covering map and the other is the set of exceptional orbits and fixed points. higher dimensional branched twist spins can be defined from both ways, however two definitions may be different from each other.

Let us consider the following n_2-fold covering map

$$(M^5, \tau_{m_2,n_2}(\tau_{m_1,n_1}(K))) \to (S^5, \tau_{m_2}(\tau_{m_1,n_1}(K))).$$

Then it is not known the condition that M is diffeomorphic to S^4. To study such conditions, the results about the classification of 5-manifolds and the geometric structure about the fibration of 4-manifolds over S^1 are needed. On the other hand, the classication of circle actions on S^5 is open problem, that is called Montgomery-Yang problem. Thus the applicant will first examine a 3-knots obtained by a concrete circle action and study under the assumption that M is diffeomorphic to S^5. From that, he will observe the relationship between geometric structures of high dimensional manifolds and high dimensional knots.

References

[1] M. Fukuda and M. Ishikawa, *Twist spun knots of twist spun knots of classical knots*, arXiv:2409.00650.