Our studies (Naoki Kitazawa) We have studied theory of Morse functions and higher dimensional versions as a study of "Singularity theory of differentiable maps" and "Applications to geometry of manifolds". For example, constructing such maps is difficult and we have constructed. We present some studies.

- On differentiable manifolds Morse functions exist densely. Singular points know homology groups and homotopy, information on deformation. Such theory is generalized to higher dimensions. Thom and Whitney started, Levine and later Eliashberg have studied more, and recently Saeki and Sakuma have been studying actively. The class of fold maps is a class of smooth maps locally represented as the product maps of Morse functions and identity maps. For example, the canonical projection of the unit sphere gives a simplest example with its set of singular points being the equator and its image is an embedded sphere. We have introduced round fold maps as generalizations, investigated fundamental invariants of the manifolds, and constructed non-trivial examples on elementary manifolds such as connected sums of total spaces of bundles over spheres whose fibers are spheres (Articles 1.1、 1.2、 2.1、 3). We have also classified codimension -1 round fold maps (Article 1.5). We have also checked that a 3-dimensional closed, and orientable manifold admits such a map into the plane if and only if it is a so-called graph manifold (Article 1.7).

- Morse functions with exactly two singular points on homotopy spheres and the canonical projections of the unit spheres are generalized to special generic maps. In the story presented above, Burlet and de Rham defined such maps in 1970s and Saeki and Sakuma have been studying actively. These maps restrict the topologies and the differentiable structures strongly. For example, elementary manifolds above admit natural special generic maps in considerable cases. We have investigated cohomology rings of manifolds admitting such maps (Articles 4.1-2、 4.4-7、 4.17). For example, we have seen non-existence of such maps for projective spaces.

- The Reeb graph of a smooth function is the quotient map consisting of all connected components of preimages. In the case of smooth functions on closed manifolds with the sets of all critical values being finite, they are naturally graphs (2022 Saeki), for example. In the 20th century, it has appeared and it simplifies the manifold. Such graphs are also important in visualization. In 2006 Sharko has asked "can we have nice smooth functions whose Reeb graphs are given graphs". Sharko, followed by Saeki and his student Masumoto and their refinement, has constructed nice smooth functions on closed manifolds whose Reeb graphs are given graphs. Later, Michalak has constructed Morse functions such that connected components of preimages with no critical points are spheres for suitable graphs. We have studied cases where topologies of connected components of preimages are prescribed manifolds which may not be spheres or compact (Articles 1.3、 1.4、 1.8、 4.3). As a challenging case, we have studied real algebraic cases (Articles 1.6、 2.1、 4.10-12).