Research results (Mikiya Masuda) January, 2025

Around 2000, I got an idea to develop toric geometry, which is a bridge connecting algebraic geometry and combinatorics, from the viewpoint of topology, and started working along the idea with help of Professor Akio Hattori. On the other hand, almost around the same time, Buchstaber and Panov in Moscow was developing toric geometry from the viewpoint of topology independently from us. I learned it when I met Taras Panov at a conference in Poland in 2001. I invited him to Japan many times and developed toric topology jointly with him. Through this joint work we found new viewpoint and problems which were not in toric geometry. Since then, many more people joined to this area so that this area developed rapidly and has become a new mathematical subject called toric topology.

Some of my main contribution to toric topology are as follows.

(1) Closed smooth 2n manifolds with smooth actions of compact torus of dimension n are called torus manifolds (precisely speaking, we assume the existence of a fixed point and give a condition on orientation. Compact smooth toric varieties (which we briefly call toric manifolds) can be thought of as torus manifolds by restricting actions of $\mathbb{C}^*$ torus to S^1 torus. One can associate multi-fans, which is a generalization of fan, to torus manifolds and using this one can develop toric geometry from the viewpoint of topology and extend toric geometry. However, this correspondence from torus manifolds to multi-fans is not injective. Later, Ishida, Fukukawa and I introduced the notion of topological toric manifolds and showed there is a one-to-one correspondence between topological toric manifolds and what is called topological fans. This is a generalization of the fundamental theorem in toric geometry in the compact and smooth case.

(2) The celebrated g-theorem characterizes the number of faces of simplicial convex polytopes by three conditions. The necessity of these three conditions was proved by Stanley using toric geometry in 1980's. The boundary of simplicial convex polytopes provides triangulations of spheres. Simplicial poset is a notion which weakens a condition on triangulation. It corresponds to simplicial cell decomposition in topology. Using an idea of toric topology, I characterized the number of faces of simplicial cell decompositions of spheres.

(3) Classification of toric manifolds as varieties reduce to classification of fans. However, classification of toric manifolds as smooth manifolds is unknown. Concerning this, I proposed what is called the cohomological rigidity problem (for toric manifolds) which asks whether cohomology ring distinguishes toric manifolds. Many partial affirmative results are obtained for this problem but no counterexamples are known.

(4) Small cover is a closed smooth n-manifold M with $(\mathbb{Z}/2)^n$-action whose orbit space is a simple polytope P. In other works, M can be constructed by gluing 2n copies of P along their boundaries. Small covers are aspherical manifolds in many cases. In particular, when P is a Pogorelov polytope, which can be realized in Lobachevsky with right angle, the corresponding small covers are compact hyperbolic 3-manifolds. Buchstaber, Erokhovets, Panov, Park and I showed that these hyperbolic 3-manifolds are distinguished by their cohomology rings with $\mathbb{Z}/2$-coefficients.