Background Vertex operator algebras (VOA) are introduced as axiomatizations of the energy-momentum tensor in conformal field theory (the generating operator of Virasoro algebras), primary fields and their descendant fields, operator product expansions of fields, and so on. As in the theory of algebra, VOA modules can be defined, and each module admits generalised eigenvalue decompositions by the zero mode L_0 of the Virasoro algebra. Since the operator product expansion of a field consists of an infinite number of Fourier modes, it is generally difficult to investigate the simplicity of VOA modules and the property of morphisms between modules.

Given a vertex operator algebra V and a V-module M, and a zero mode L_0 acting on M with Jordan blocks, we call the module M a logarithmic module and V a logarithmic VOA. The adjective "logarithmic" corresponds to the fact that log-type solutions appear in conformal field theory for solutions of holonomic systems of regular singularity type. On the physics side, log-type solutions describe interesting phenomena such as polymers, spin chains, percolation, and the sandpile models.

In the theory of vertex operator algebras, the structure of the abelian and tensor categories of VOA modules is important. For non-logarithmic modules of vertex operator algebras, the structure of vector spaces and tensor products are conventionally determined by studying the characters of representations. On the other hand, for logarithmic modules, Virasoro's zero mode L_0 acts non-semisimply, making it difficult to determine the structure by using the characters. Therefore, the study of modules of logarithmic vertex operator algebras is more challenging.

Famous examples of logarithmic vertex operator algebras are the family of vertex operator algebras called triplet W-algebras and the non-unitary $N = 2$ Virasoro superalgebra. The former are vertex operator algebras satisfying a certain finiteness condition called C_2 cofinite condition, and are famous for their connection with the representation theory of quantum groups at the root of unity (Feigin *et al.* , *Nucl. Phys.B*, 2006 & Adamović and Milas. *Math. Phys*, 2009). The latter belong to a family of W-algebras called principal W-algebras, which are associated to vertex operator algebras of affine sl_2 through the cosset construction (Creutzig *et al.* , *JHEP*, 2019). Just as the irreducible decomposition of tensor products was important in the representation theory of the algebra, in these vertex operator algebras the structure of tensor products (or fusion products) between modules has been studied from various aspects of physics, and further mathematical refinement of the theory is expected.

Results In our previous studies of triplet W-algebras and $N = 2$ Virasoro superalgebras, we have discovered a certain deformation method for vertex operators. We call it the ϵ deformation method for vertex operators. Using the ϵ deformation technique, we have revealed the following properties.

(1) For the supertriplet W-algebra $\mathcal{SW}(m)$ introduced by Adamovi'c and Milas, we determined the structure of all projective modules and the fusion product between simple and projective modules. Furthermore, we proved that the tensor category of the $\mathcal{SW}(m)$-modules is rigid (Nakano., arXiv:2412.20898, 2024).

(2) We determined the structure of the fusion product between the weight modules of $N = 2$ Virasoro superalgebras and proved that the tensor category of the weight modules is rigid (Nakano, Orosz Hunziker, Ros Camacho and Wood. arXiv:2411.11387, 2024 2024).

The result on (2) was achieved at the same time by Creutzig, McRae, and Yang by a different proof method (arXiv:2411.11386).

Intuitively, the above deformation method can be described as a renormalization of the representation by the small parameter ϵ. The important points of the ϵ deformation method can be described as follows.

a By renormalizing the small parameter ϵ to the vertex operators, we can use the method of free field realizations with free bosons and free fermions.

b Then, by analytical evaluation with respect to ϵ, we can construct the intertwining operators and N-point correlation functions, which are important in the representation theory of VOA.

This method of ϵ deformation is expected to be applicable to the theory of various logarithmic vertex operators.