Future Research Plan

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I will continue research on canonical metrics and stability of polarized algebraic manifolds. Here are two specific research themes:

Determination of Uniform Relative K-stability and Relative K-instability

For polarized toric manifolds, determining uniform relative K-stability or relative K-instability of given polarized toric manifolds is equivalent to determining the existence or non-existence of Calabi's extremal metrics [3]. The goal is to completely determine the stability for three-dimensional toric Fano manifolds (with respect to anticanonical polarization). Among the 18 three-dimensional toric Fano manifolds, 13 are known to be uniformly relatively K-stable, but the status of the remaining 5 is unknown. The research aims to discover new criteria for uniform relative K-stability and relative K-instability, including methods based on specific geometric features such as projective bundles and blow-ups. Recently, it was shown that relatively K-unstable toric Fano manifolds exist in all dimensions ten and above. I am also interested in finding relatively K-unstable examples in lower dimensions, including cases with singularities.

Research on Hyperplane Sections of Segre Varieties

Let X be a smooth hypersurface of bidegree $(1,1)$ in $\mathbf{P}^m \times \mathbf{P}^n$, which is also a smooth hyperplane section of the Segre variety. In the 1980s, Sakane and Hano proved that when $m \neq n$, there exist no constant scalar curvature Kähler metrics in any Kähler class on X. Inspired by this result, I proved that when $m \neq n$, the Futaki invariant of X is non-zero for any rational Kähler class. This naturally leads to the question: whether or not X admits canonical metrics such as Kähler-Ricci solitons, Mabuchi solitons, or extremal metrics? The research will begin with calculating the Mabuchi constant of X with respect to anticanonical polarization. Computing the δ-invariant of X is also interesting. When $m = n$, X admits homogeneous Kähler-Einstein metrics, but the existence of constant scalar curvature Kähler metrics in general Kähler classes remains to be determined. For smooth hypersurfaces in $\mathbf{P}^m \times \mathbf{P}^n$ of higher bidegree, the existence of Kähler-Einstein metrics is unknown even when $m = n$ (except for the cases $m = n = 3$ with bidegrees $(1,2)$ and $(1,3)$ treated in [1]), warranting investigation from both differential-geometric and algebro-geometric perspectives.

References

[1] J. Cable, Kähler-Einstein metrics on symmetric general arrangement varieties. *Manuscripta Math.* **168** (2022), pp.119–135.

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[3] Y. Nitta and S. Saito, A uniform version of the Yau-Tian-Donaldson correspondence for extremal Kähler metrics on polarized toric manifolds. arXiv:2110.10386.

[4] S. Saito, K-instability of hyperplane sections of Segre varieties. arXiv:2407.12412.