Summary of Research Achievements **Zengo Tsuboi**

In my master's thesis, I provided a general theory for obtaining the equations satisfied by the parameters of higher-order decomposition of exponential operators, using Lyndon words, which correspond one-to-one with the basis of the free Lie algebra [1][1]. During my doctoral course, I studied the analytic Bethe ansatz (BA) and the T-system, which is a functional equation satisfied by the transfer matrix of solvable models. First, I obtained the solutions to the T-system of type D [2] and the solutions to the discrete affine Toda field equation, which can be obtained by extending the T-system [3]. Next, I developed methods for applying the BA to models associated with superalgebras [4-8] and derived [9-11, 15] the TBA equations, which provide the finite-temperature free energy of integrable quantum spin chains, using the T-system proposed in [8]. These results are expected to be useful for research in particle physics and condensed matter physics.

I also studied the Fermionic formulas and Q-systems derived from the discussion on the completeness of the Bethe ansatz [12-14]. Furthermore, I provided Casorati determinant solutions to the T-system [16]. These contribute to the development of mathematics (especially the representation theory of quantum affine algebras) from a physical perspective.

The TBA equations, which provide the finite-temperature free energy of integrable quantum spin chains, are nonlinear integral equations with an infinite number of variables. In view of this, I systematically derived integral equations that contain only the number of variables equal to the rank, equivalent to the TBA equations associated with algebras of general rank [17, 18, 20, 21, 23]. The application of these equations to the calculation of thermodynamics of spin ladder systems has been found to be in good agreement with experimental results in condensed matter physics [19, 25]. Although calculating correlation functions of solvable models is a difficult problem, I succeeded in evaluating the high-temperature expansion of correlation functions of the XXX model [22, 24].

I constructed the Baxter Q-operator for $U_q(\hat{sl}(2|1))$ using infinite-dimensional representations of the q-oscillator algebra in a form applicable to both lattice models and CFT [26]. Furthermore, I obtained Wronskian-type solutions to the T-system for $U_q(\hat{gl}(M|N))$ in relation to Baxter Q-operators [27]. In particular, that there can be 2^{N+M} types of Baxter Q-operators was pointed out for the first time. By applying the techniques and concepts proposed in [27], I provided solutions to the T-system and Q-system proposed in relation to the AdS/CFT correspondence [28, 29, 31]. I proposed a new method for constructing Baxter Q-operators using derivatives on groups and derived the Bethe equations without using the traditional Bethe ansatz [30]. Additionally, I proposed the Master T-operator, which corresponds to the tau function [32, 33]. The Hirota-type bilinear identities satisfied by the Baxter Q-operators and T-operators are systematically derived from this Master T-operator. Using the properties of the universal R-matrix, I proved the factorization formula for the L-operator with respect to the Baxter Q-operator for the Verma module of $U_q(\hat{sl}(2))$ [34]. I studied the asymptotic representations of $U_q(\hat{gl}(M|N))$ and obtained solutions to the Yang-Baxter equations related to Baxter Q-operators using q-oscillator algebra (L-operators) [35]. Furthermore, in [41], I considered the realization of the Verma module of $U_q(\hat{gl}(M|N))$ using q-oscillator algebra and provided a general realization of the contracted algebra discussed in [35] through contraction involving the q-oscillator algebra. I extended the results of [32] to the case of $gl(N|M)$ [36]. I showed that the zeros of the Master T-operator are described by the equations of motion of the Ruijsenaars-Schneider model. Additionally, I proposed an algebraic equation that gives the eigenvalues of the Hamiltonian of supersymmetric spin chains.

In relation to the AdS/CFT correspondence, I proposed a factorization formula for Beisert's S-matrix (with respect to the R-matrix related to the free fermion model) [37].

I solved the intertwining relations of the augmented q-Onsager algebra and obtained a general K-operator (involving elements of the Cartan subalgebra of $U_q(sl(2))$) that provides solutions to the reflection equation [39]. I also considered the asymptotic representations of this algebra in the Verma module and provided the K-operator necessary for constructing the Baxter Q-operator for boundary models. Furthermore, I generalized the K-operator proposed in [39] to the cases of $U_q(\hat{gl}(N))$ [40] and triangular q-Onsager algebra [42]. In [43], I provided the Baxter TQ-relations for the case where the quantum space is general, using the K-operator given in [39] and the universal R-matrix.

I provided solutions to the set-theoretic (quantum) Yang-Baxter equations as products of quasi-Plücker coordinates related to matrices constructed from L-operators, given as images of the universal R-matrix of $U_q(gl(n))$ [38]. Furthermore, by taking the semiclassical limit, I provided a new determinant representation of the (classical) Yang-Baxter map.

It is known that there is a certain correspondence between the representations of twisted quantum affine superalgebras and untwisted quantum affine superalgebras. In view of this, I provided QQ-relations and Wronskian-type eigenvalue formulas for the transfer matrix for various superalgebras as reductions (a kind of folding) of those associated with $U_q(gl(M|N)^{(1)})$ [31, 44, 45].

[1]Refer to the number in the "List of Publications"