Ribbonness of Kervaire's sphere-link in homotopy 4-sphere and its consequences to 2-complexes

Akio KAWAUCHI

Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University

Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

kawauchi@omu.ac.jp

ABSTRACT

M. A. Kervaire showed in 1965 that every finitely presented group with deficiency k, weight k and the free first homology of rank k is the fundamental group of a smooth sphere-link of component k in a smooth homotopy 4-sphere. Use the smooth unknotting conjecture and the smooth 4D Poincaré conjecture. Any such 2-link is shown to be equivalent to a sublink of a free ribbon sphere-link in the 4-sphere, whose ribbon disk-link complement in the 4-disk is also shown to have a finite aspherical 2-complex as a spine. Every ribbon sphere-link in the 4-sphere is also shown to be a sublink of a free ribbon sphere-link in the 4-sphere, so that the complement of every ribbon disk-link in the 4-disk has a finite aspherical 2-complex as a spine. This implies that every subcomplex of a finite contractible 2-complex is aspherical (partially yes for J. H. C. Whitehead's conjecture).

Keywords: Kervaire's sphere-link, ribbon sphere-link , 2-complex, J. H. C. Whitehead's conjecture

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1. Introduction

A finitely presented group G has *deficiency* k if G has a finite presentation

$$< x_1, x_2, ..., x_n | \, r_1, r_2, ..., r_s >$$

with $k = n - s$, and has *weight* k if there are k elements $y_1, y_2, ..., y_k$ in G whose normal closure is equal to G. The elements $y_1, y_2, ..., y_k$ are called a *k-weight system* of G.

A smooth homotopy 4-sphere is a smooth 4-manifold M homotopy equivalent to the 4-sphere S^4. A *smooth surface-link* in M is the image K of a smooth embedding from a closed oriented surface F into M. If K consists of only 2-spheres, then K is called an *S^2-link* in M. A *legged k-loop system with base point v* is a graph $\vee m$ consisting of a loop system $m = \{m_1, m_2, \ldots, m_k\}$ and a path system $\omega = \{\omega_1, \omega_2, \ldots, \omega_k\}$ such that the leg ω_i connects the base point v and a point $p_i \in m_i$ for every i. A *meridian system* of an S^2-link K with k components in a smooth homotopy 4-sphere M is a legged k-loop system $\vee m$ with base point v which is embedded in $M \setminus L$ and whose loop system m consists of a meridian loop of every component of K. M. A. Kervaire showed the following theorem in [11].

Kervaire's Theorem. If a finitely presented group G has deficiency k, weight k and the first homology $H_1(G) = G/[G,G] \cong \mathbf{Z}^k$ (the rank k free abelian group), then there is an S^2-link K with k components in a smooth homotopy 4-sphere M such that there is an isomorphism $G \cong \pi_1(M \setminus K, v)$ sending the k-weight system to a meridian system of K.

Kervaire's construction of an S^2-link to obtain this theorem is explained as follows:

Kervaire's construction of an S^2-link. For a presentation $< x_1, x_2, ..., x_n | \, r_1, r_2, ..., r_{n-k} >$ and a weight system $y_1, y_2, ..., y_k$ of the group G, Let $P(n; n-k, k)$ be the *triple system* of the free group $< x_1, x_2, ..., x_n >$, the relator system $r_1, r_2, ..., r_{n-k}$ written as words in $x_1, x_2, ..., x_n$ and a weight system $y_1, y_2, ..., y_k$ written as words in $x_1, x_2, ..., x_n$. Identify the free group $< x_1, x_2, ..., x_n >$ with the fundamental group $\pi_1(Y, v)$ of the 4D closed handlebody $Y = S^4 \$_{i=1}^n S^1 \times S^3_i$ of genus n by taking x_i to be a homotopy class of the standard loop $S^1 \times \mathbf{1}_i$ of the product summand $S^1 \times S^3_i$ for all i. Let X be the 4-manifold obtained from Y by surgery along tubular neighborhoods $\ell(r_1) \times D^3, \ell(r_2) \times D^3, \ldots, \ell(r_{n-k}) \times D^3$ of simple loops $\ell(r_1), \ell(r_2), \ldots, \ell(r_{n-k})$ in Y representing the words $r_1, r_2, \ldots, r_{n-k}$ in $\pi_1(Y, v)$. The fundamental group $\pi_1(X, v)$ is identified with the group G with the presentation $< x_1, x_2, ..., x_n | \, r_1, r_2, ..., r_{n-k} >$ by van Kampen theorem. Let $m\omega$ be a legged k-loop system with base point v embedded in X which represents the weight system $y_1, y_2, \ldots, y_k$ of $\pi_1(X, v)$. Let M be the 4-manifold obtained by surgery along tubular neighborhoods $m_i \times D^3 \, (i = 1, 2, \ldots, k)$ of the loop system $m_i \, (i = 1, 2, \ldots, k)$ in X. The manifold M is a smooth homotopy 4-sphere and for the S^2-link K of components $K_i = p_i \times \partial D^3 \, (i = 1, 2, \ldots, k)$ in M, the fundamental group $\pi_1(M \setminus K, v)$ is isomorphic to the group G, where the legged k-loop system $\vee m$ is a meridian system of K in M. This completes Kervaire's construction of a desired S^2-link.

In this construction, note that the S^2-link K obtained is uniquely determined by the triple system $P(n; n-k, k)$ of the free group $< x_1, x_2, ..., x_n >$, the relator system $r_1, r_2, ..., r_{n-k}$ and a weight system $y_1, y_2, ..., y_k$. This S^2-link K is called *Kervaire's sphere-link* of *$P(n; n-k, k)$-type* or simply an *S^2-link of $P(n; n-k, k)$-type*.

A smooth surface-link L in S^4 is a *trivial* surface-link if the components of L bound disjoint handlebodies smoothly embedded in S^4. The fundamental group $\pi_1(S^4 \setminus L, v)$ is a *meridian-based free group* if $\pi_1(S^4 \setminus L, v)$ is a free group with a basis represented by a meridian system of L with base point v.

The purpose of this paper is to research Kervaire's theorem carefully by using

Smooth Unknotting Conjecture for an S^2-link and Smooth 4D Poincaré Conjecture. These conjectures are stated as follows:

Smooth Unknotting Conjecture. Every smooth surface-link F in S^4 with a meridian-based free fundamental group $\pi_1(S^4 \setminus F, v)$ is a trivial surface-link.

Smooth 4D Poincaré Conjecture. Every 4D smooth homotopy 4-sphere M is diffeomorphic to S^4.

The positive proofs of the smooth unknotting conjecture and the smooth 4D Poincaré conjecture are claimed in [4, 5, 6] and [7], respectively. *From now on, every smooth homotopy 4-sphere M is identified with the 4-sphere S^4.*

An S^2-link L in S^4 is a *ribbon* S^2-link if L is equivalent to an S^2-link obtained from a trivial S^2-link O in S^4 by surgery along embedded 1-handles on O (see [10, 13] for earlier concept of ribbon surface-link).

An S^2-link K in S^4 is a *subfree ribbon S^2-link of rank n* if K is a sublink of a ribbon S^2-link L in S^4 such that the fundamental group $\pi_1(S^4 \setminus L, v)$ is a free group (not necessarily meridian based) of rank n. Then the ribbon S^2-link L has n components. By definition, a subfree ribbon S^2-link of any rank n is a ribbon S^2-link.

A main point of this paper is the following theorem.

Theorem 1.1. The following three statements on a k-component S^2-link K in the 4-sphere S^4 are mutually equivalent:

(1) The S^2-link K is an S^2-link of $P(n; n-k, k)$-type for every n greater than a constant.

(2) The S^2-link K is a k-component subfree ribbon S^2-link of rank n for every n greater than a constant.

(3) The S^2-link K is a k-component ribbon S^2-link obtained from the n-component trivial S^2-link by surgery along embedded 1-handles for every n greater than a constant.

This theorem means that there is a positive integer n such that the S^2-knot K has the properties (1), (2) and (3) for this n at the same time. The proof of Theorem 1.1 is given in Section 2. By combining Kervaire's Theorem with Theorems 1.1, the following characterization of the fundamental group $\pi_1(S^4 \setminus K, v)$ of a ribbon S^2-link K in S^4 is obtained.

Corollary 1.2. A group G has a finite presentation with deficiency k and weight k and the first homology $H_1(G) \cong \mathbf{Z}^k$ if and only if there is an isomorphism $G \to \pi_1(S^4 \setminus K, v)$ sending the weight system to a meridian system for a ribbon S^2-link K of k components.

It is a standard fact that every ribbon S^2-link L in S^4 is regarded as the double of a ribbon disk-link L^D in the 4-disk D^4 (see [10, II]). The *compact complement of a ribbon disk-link L^D in the 4-disk D^4* is the compact 4-manifold $E(L^D) = \mathrm{cl}(D^4 \setminus N(L^D))$ for a regular neighborhood $N(L^D)$ of L^D in D^4. The following theorem can be shown by considering a ribbon disk-link K^D as the ribbon disk-link K^D of a subfree ribbon S^2-link K of some rank n .by using Theorem 1.1.

Theorem 1.3. The compact complement $E(L^D)$ of every ribbon disk-link L^D in the 4-disk D^4 is homotopy equivalent to an aspherical 2-complex.

The result of the case of a ribbon disk-knot in D^4 has been conjectured by Howie [2] after having found some gaps on the arguments of Yanagawa [14] and Asano, Marumoto, Yanagawa [1]. The proof of Theorem 1.3 is done in Section 3. For the proof two claims are provided. One claim is that the second homotopy group

$$\pi_2(D^4 \setminus L^D, v) = 0$$

for the ribbon disk-link K^D in the 4-disk D^4 of a subfree ribbon S^2-link K of any rank n in S^4 (see Lemma 3.1 for the proof). The other claim is that the compact complement $E(L^D)$ of a ribbon disk-link L^D in D^4 is homotopy equivalent to a 2-complex (see Lemma 3.2 for the proof). For the ribbon disk-link L^D in D^4 of every ribbon S^2-link L in S^4, the inclusion homomorphism

$$\pi_1(D^4 \setminus L^D, v) \to \pi_1(S^4 \setminus L, v)$$

is shown to be an isomorphism (see Lemma 3.3 for the proof). Since the fundamental group of a finite-dimensional aspherical complex is torsion-free, the following corollary is obtained.

Corollary 1.4. The fundamental group $\pi_1(S^4 \setminus L, v)$ of every ribbon S^2-link in the 4-sphere S^4 is torsion-free.

This result answers positively an old question in [10, II(pp.57-58)]. Theorem 1.3 is closely related to J. H. C. Whitehead Conjecture [12] stated as follows (cf. [2]).

J. H. C. Whitehead Conjecture. Every connected subcomplex of an aspherical 2-complex is aspherical.

The following result is obtained from Theorem 1.3.

Corollary 1.5. Every connected subcomplex of a contractible finite 2-complex is aspherical.

The proof of Corollary 1.5 is given in Section 3. For general references on this paper, see [3].

2. Proof of Theorem 1.1

Let X be a closed connected oriented smooth 4-manifold, and k a loop system of disjoint simple loops $k_i \, (i = 1, 2, \ldots, n)$ in X. A *surgery along the loop system k* is a replacement operation of a normal 3-disk bundle system $k_i \times D^3 \, (i = 1, 2, \ldots, n)$ of $k_i \, (i = 1, 2, \ldots, n)$ in X by a 2-disk bundle system $D^2_i \times S^2 \, (i = 1, 2, \ldots, n)$ of the 2-sphere system $K_i = 0_i \times S^2 \, (i = 1, 2, \ldots, n)$ under the identification that $\partial D^2_i = k_i \, (i = 1, 2, \ldots, n)$ and $\partial D^3 = S^2$. The sphere system $K_i \, (i = 1, 2, \ldots, n)$ form an S^2-link K in the smooth 4-manifold X' resulting from X by this surgery. The 4-manifold X' is said *to be obtained from the 4-manifold X by surgery along a loop system k* in X, and conversely the 4-manifold X is said *to be obtained from the 4-manifold X' by* *surgery along a sphere system* K in X'. Note that there are canonical fundamental group isomorphisms

$$\pi_1(X, v) \cong \pi_1(X \setminus k, v) \cong \pi_1(X' \setminus K, v)$$

by general position (see [9, Lemma 3.1]). The proof of Theorem 1.1is done as follows.

Proof of Theorem 1.1.

Proof of (1)→(2). The S^2-link K of $P(n; n-k, k)$-type in S^4 for any n is constructed from the triple system consisting of the free basis $x_i\,(i = 1, 2, \ldots, n)$, the relator system $r_i\,(i = 1, 2, \ldots, n-k)$ written as words in $x_i\,(i = 1, 2, \ldots, n)$ and a weight system $y_j\,(j = 1, 2, \ldots, k)$ written as words in $x_i\,(i = 1, 2, \ldots, n)$. Identify the free group $< x_1, x_2, ..., x_n >$ with the fundamental group $\pi_1(Y, v)$ of the closed handlebody $Y = S^4\$_{i=1}^n S^1 \times S^3_i$ of rank n. Note that the elements $r_i, y_j\,(i = 1, 2, \ldots, n-k; j = 1, 2, \ldots, k)$ normally generate the free group $\pi_1(Y, v)$. Represent the elements $r_i, y_j \in \pi_1(Y, v)\,(i = 1, 2, \ldots, n-k; j = 1, 2, \ldots, k)$ by a disjoint simple loop system $k(r_i), k(y_j)\,(i = 1, 2, \ldots, n-k; j = 1, 2, \ldots, k)$ in Y. The 4-manifold M obtained from Y by surgery along the loop system $k(r_i), k(y_j)\,(i = 1, 2, \ldots, n-k; j = 1, 2, \ldots, k)$ is a smooth homotopy 4-sphere identified with S^4. Let L be the S^2-link in $M = S^4$ of the sphere system $K(r_i), K(y_j)\,(i = 1, 2, \ldots, n-k; j = 1, 2, \ldots, k)$ occurring from the loop system $k(r_i), k(y_j)\,(i = 1, 2, \ldots, n-k; j = 1, 2, \ldots, k)$ by surgery. The fundamental group $\pi_1(S^4 \setminus L, v)$ is a free group (although it is not always meridian-based free). By [9, Lemma 3.4], it is shown that the S^2-link L is a free ribbon S^2-link in S^4. The sublink K of L consisting of $K(y_j)\,(j = 1, 2, \ldots, k)$ is a subfree ribbon S^2-link in S^4 of rank n. By adding trivial S^2-components to L, the positive integer n is replaced by any positive integer greater than a constant. This shows (1)→(2).

Proof of (2)→(1). Let K be a k-component sublink of an n-component S^2-link L in S^4 such that the fundamental group $\pi_1(S^4 \setminus L, v)$ is a free group of rank n. Let Y be the 4-manifold obtained from S^4 by surgery along L. Then Y is diffeomorphic to the 4D closed handlebody $S^4\$_{i=1}^n S^1 \times S^3_i$ of genus n, which is shown in [9, Lemma 3.2] by using Smooth Unknotting Conjecture for an S^2-link and Smooth 4D Poincaré Conjecture. Since the fundamental group $\pi_1(S^4 \setminus L, v)$ is identified with the fundamental group $\pi_1(Y, v)$, the S^2-link K is an S^2-link of $P(n; n-k, k)$-type consisting of the free group $\pi_1(Y, v) =< x_1, x_2, \ldots, x_n >$, a relator system $r_1, r_2, ..., r_{n-k}$, coming from the meridians of $L \setminus K$, and a weight system $y_1, y_2, ..., y_k$ coming from the meridians of K. By adding trivial S^2-components to L, the positive integer n is replaced by any positive integer greater than a constant. This shows (2)→(1).

Proof of (2)→(3). This proof is trivial since a sublink of a ribbon S^2-link is a ribbon S^2-link and any ribbon S^2-link is obtained from the n-component trivial S^2-link by surgery along 1-handles for any positive integer n greater than a constant.

Proof of (3)→(2). The k-component ribbon S^2-link K is obtained from an n-component trivial S^2-link O in S^4 by surgery along a 1-handle system h on O. Let $O \times I$ be a collar of O in S^4, and $W = O \times I \cup h$ a k-component compact 3-manifold bounded by $K \cup (-O)$. Let $K_i\,(i = 1, 2, \ldots, k)$ be the components of K. Let O' be a $(n-k)$-component sublink of O obtained by removing any one component of O used for the surgery to obtain the component K_i for every i. Then there are isomorphisms

$$\pi_1((S^4 \setminus W, v) \to \pi_1(S^4 \setminus K \cup O'), v) \quad \text{and} \quad \pi_1(S^4 \setminus W, v) \to \pi_1(S^4 \setminus O, v).$$

In fact, the first isomorphism is established by using that every point in the interior of W can be pushed out of W through the S^2-system $\partial W \setminus O'$ without touching the S^2-link $K \cup O'$ and that every intersection loop between W and an immersed disk whose boundary loop is in the complement $S^4 \setminus K \cup O'$ can be shortened into a point in the interior of W because W is simply connected. The second isomorphism is established by using that there is a strong deformation retract from the 4-manifold W into the union of O and some panning arcs because the presence of the arcs does not affect the fundamental group isomorphism. This means that the n-component S^2-link $K \cup O'$ is a free ribbon S^2-link of rank n and thus, K is a subfree S^2-link of rank n. The positive integer n is replaced by any positive integer greater than a constant. This shows (3)→(2).

This completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.3

A *trivial proper disk system* in the 4-disk D^4 is a proper disk system $d_i\,(i = 1, 2, \ldots, n)$ in D^4 obtained by an interior push of a disk system $d_i^0\,(i = 1, 2, \ldots, n)$ in the 3-sphere $S^3 = \partial D^4$. A k-component ribbon disk-link in D^4 is a proper disk system $\cup_{i=1}^n d_i \cup_{j=1}^{n-k} b_j$ in D^4 obtained by an interior push of a disk union $\cup_{i=1}^n d_i \cup_{j=1}^{n-k} b_j^0$ in D^4 which is a union of a trivial disjoint k disk system $d_i\,(i = 1, 2, \ldots, n)$ in D^4 and a disjoint band system $b_j^0\,(j = 1, 2, \ldots, n-k)$ in S^3 spanning the trivial loop system $\partial d_i = \partial d_i^0\,(i = 1, 2, \ldots, n)$. By construction, a k-component ribbon S^2-link L in S^4 is the double of a k-component ribbon disk-link L^D in D^4 (see [10, II]). Let α be the reflection of the ribbon S^2-link (S^4, L) changing one copy (D^4, L^D) and the other copy $(-D^4, -L^D)$. In the proof of the following lemma, Theorem 1.1 is essentially used.

Lemma 3.1. For the ribbon disk-link K^D in the 4-disk D^4 of a subfree ribbon S^2-link K of rank n in S^4, the second homotopy group $\pi_2(D^4 \setminus L^D, v) = 0$.

Proof of Lemma 3.1. Since K is a subfree ribbon S^2-link in S^4 of rank n, take an n-component ribbon S^2-link L with the fundamental group $\pi_1(S^4 \setminus L, v)$ a free group and with K as a sublink. Let K^D be the ribbon disk-link of K in D^4, and L^D the ribbon disk-link of L in D^4 containing K^D as a sublink. Let $\tilde{S}$ be an immersed 2-sphere in the compact exterior $E(K^D)$, which is considered as an immersed 2-sphere in the compact exterior $E(L^D)$ by taking the ribbon disk system $L^D \setminus K^D$ in a thin boundary collar of D^4. Let Y be the 4-manifold obtained from S^4 by surgery along L. Since $\pi_1(Y, v) \cong \pi_1(S^4 \setminus L, v)$ is a free group of rank n, the 4-manifold Y is identified with the 4D closed handlebody $S^4 \#_{i=1}^n S^1 \times S_i^3$ of genus n by [9, Lemma 3.2]. Hence the immersed sphere $\tilde{S}$ in $E(L^D)$ bounds an immersed 3-ball $\tilde{B}$ in Y. Let $k(L)$ be the loop system in Y occurring from the surgery along L. By general position, the loop system $k(L)$ meets transversely the immersed 3-ball $\tilde{B}$ in a finite set, say an s point set. Then there is a compact s-punctured immersed 3-ball $\tilde{B}^{(s)}$ in the compact exterior $E(L)$ of L in S^4 such that $\partial \tilde{B}^{(s)} \supset \tilde{S}$ and $\partial \tilde{B}^{(s)} \setminus \tilde{S}$ is a 2-sphere system

$S_i\,(i = 1, 2, \ldots, s)$ in the boundary $\partial E(L)$. Note that the compact exterior $E(L)$ is the union of the compact exterior $E(L^D)$ and the other copy $E(-L^D)$ changing by the reflection α. By transforming the intersection part $\tilde{B}^{(s)} \cap E(-L^D)$ into $E(L^D)$ by the reflection α, the punctured immersed 3-ball $\tilde{B}^{(s)}$ is taken in the compact exterior $E(L^D)$ so that the (possibly singular) 2-spheres $S_i\,(i = 1, 2, \ldots, s)$ are in $L^D \times S^1 \subset \partial E(L)$. Since each component of $L^D \times S^1$ is aspherical, the immersed 2-speres $S_i\,(i = 1, 2, \ldots, s)$ bounds singular 3-balls in $L^D \times S^1$. This means that the immersed sphere $\tilde{S}$ is null-homotopic in $E(L^D) \subset E(K^D)$. Hence, $\pi_2(E(K^D), v) = 0$. *square*

Although the following lemma is more or less known (cf. [1]), the proof is given here for convenience.

Lemma 3.2. The compact complement $E(L^D)$ of a k-component ribbon disk-link L^D in the 4-disk D^4 has a handle decomposition consisting of one 0-handle and n 1-handles and $n-k$ 2-handles for some n. Thus, the compact complement $E(L^D)$ is homotopy equivalent to a 2-complex.

Proof of Lemma 3.2. Assume that the k-component ribbon disk-link L^D in D^4 is given $\cup_{i=1}^{n} d_i \cup_{j=1}^{n-k} b_j$ for a trivial proper disk system $d_i\,(i = 1, 2, \ldots, n)$ and a band system $b_j\,(j = 1, 2, \ldots, n-k)$ lifting the band system $b_j^0\,(j = 1, 2, \ldots, n-k)$ in the 3-sphere $S^3 = \partial D^4$. Let $h_j\,(j = 1, 2, \ldots, n-k)$ be the 1-handle system on the disk system $d_i\,(i = 1, 2, \ldots, n)$ which is the trace of the lifting from the band system $b_j^0\,(j = 1, 2, \ldots, n-k)$ to the band system $b_i\,(i = 1, 2, \ldots, n-k)$. Then the compact exterior E of the union $\cup_{i=1}^{n} d_i \cup_{j=1}^{n-k} h_j$ in D^4 is diffeomorphic to the compact exterior $E(\cup_{i=1}^{n} d_i)$ of the disk system $d_i\,(i = 1, 2, \ldots, n)$ in D^4, which has a handle decomposition consisting of one 0-handle and n 1-handles. The compact complement $E(L^D)$ is obtained from E by adding $n-k$ 2-handles which are dual to the 1-handles $h_j\,(j = 1, 2, \ldots, n-k)$ on the disk system $d_i\,(i = 1, 2, \ldots, n)$. *square*

The proof of Theorem 1.3 is done as follows (although it is not hard after Lemmas 3.1,3.2).

Proof of Theorem 1.3. Let K^D be a ribbon disk-link in D^4 whose double is a ribbon S^2-link in S^4, which is a subfree ribbon S^2-link K by Theorem 1.1. By Lemma 3.1, the second homotopy $\pi_2(E(K^D), v) = 0$ for the compact exterior $E(K^D)$ of the ribbon disk-link K^D in D^4. Since the compact exterior $E(K^D)$ is homotopy equivalent to a 2-complex by Lemma 3.2, the universal cover $\tilde{E}(K^D)$ of $E(K^D)$ has the trivial reduced homology $\tilde{H}_*(\tilde{E}(K^D); Z) = 0$. This means that $\pi_q(E(K^D), v) = 0$ for all $q \geq 2$, namely the compact exterior $E(K^D)$ is aspherical. □

Although the following lemma used for the proof of Corollary 1.4 may also be more or less known (cf. [?]), the proof is given here for convenience.

Lemma 3.3. For the ribbon disk-link L^D in D^4 of a ribbon S^2-link L in S^4, the inclusion $(D^4, L^D) \to (S^4, L)$ induces an isomorphism

$$\pi_1(D^4 \setminus L^D, v) \to \pi_1(S^4 \setminus L, v).$$

Proof of Lemma 3.3. Use the retraction $S^4 \backslash to D^4 \backslash L^D$ obtained from the quotient by the reflection α. Then the canonical homomorphism $\pi_1(D^4 \backslash L^D, v) \to \pi_1(S^4 \backslash L, v)$ is a monomorphism. On the other hand, For the other ribbon disk-link $(-D^4, -L^D)$ in the ribbon S^2-link $(S^4.L)$, the inclusion $(\partial D^4, \partial L^D) \to (D^4, L^D)$ induces an epimorphism $\pi_1(\partial D^4 \setminus \partial L^D) \to (D^4 \setminus L^D)$, so that the canonical monomorphism $\pi_1(D^4 \setminus L^D, v) \to \pi_1(S^4 \setminus L, v)$ is also an epimorphism and thus, an isomorphism. *square*

The proof of Corollary 1.5 on J. H. C. Whitehead's Conjecture is done as follows.

Proof of Corollary 1.5. Assume that a contractible connected 2-complex P is obtained from a bouquet of circles $c_1, c_2, \ldots, c_n$ which is the 1-skelton $P^{(1)}$ of P by attaching 2-cells $e_1, e_2, \ldots, e_n$ for some n. Identify the free group $\pi_1(P^{(1)}, v)$ with $< x_1, x_2, \ldots, x_n >$ for the basis element $x_i = [c_i]\,(i = 1, 2, \ldots, n)$, and $r_1, r_2, \ldots, r_n$ the words of the attaching data of $e_1, e_2, \ldots, e_n$ to $P^{(1)}$ forming a weight system of the free group $\pi_1(P(1), v)$ since $\pi_1(P, v) = 1$. By Corollary 1.2, there is a ribbon S^2-link (S^4, L) with an isomorphism $\pi_1(S^4 \setminus L, v) \cong < x_1, x_2, \ldots, x_n >$ sending a meridian system of L to the elements $r_1, r_2, \ldots, r_n$. By Lemma 3.3, for the compact exterior $E(L^D)$ of a ribbon disk-link (D^4, L^D) of the ribbon S^2-link (S^4, L), there is an isomorphism $\pi_1(E(L^D), v) \cong < x_1, x_2, \ldots, x_n >$ sending a meridian system of L^D to the elements $r_1, r_2, \ldots, r_n$. By Theorem 1.3, the compact exterior $E(L^D)$ is homotopy equivalent to $P^{(1)}$. Let P_S be the subcomplex of the 2-complex P by attaching any subcollection $e_{i_1}, e_{i_2}, \ldots, e_{i_s}$ of the 2-cells $e_1, e_2, \ldots, e_n$ with the attaching data $r_{i_1}, r_{i_2}, \ldots, r_{i_s}$. Then the subcomplex P_S is homotopy equivalent to the exterior $E(L^D_S)$ of the ribbon disk-link L^D_S of the ribbon S^2-link L_S obtained from L by forgetting the components of L with meridian elements $r_{i_1}, r_{i_2}, \ldots, r_{i_s}$, which is aspherical by Theorem 1.3. Thus, the sublink P_S of P is aspherical. Since every connected sublink of P is such a subcomplex P_S up to additions of bouquets of circles and homotopy equivalences, it is proved that every connected subcomplex of P is aspherical. $\square$

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