Point condensation of maximizers for Trudinger-Moser inequalities on scaling parameter

Masato Hashizume[1]

Graduate School of Advanced Science and Engineering, Hiroshima University, Higashihiroshima, 739-8527, Japan

Abstract

We study asymptotic behavior of maximizers for the critical Trudinger-Moser inequalities associated with a scaling parameter. In particular, we show the point condensation of the maximizers. We also clarify the location of the peak of maximizers in the critical case, as well as in the subcritical case. The location of the peak of maximizer depends on geometric properties of a bounded domain.

Keywords: asymptotic expansion, Trudinger-Moser inequality, two dimension
2010 MSC: Primary 35B38; Secondary 35A09, 35B40, 35J15, 35J61

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. It is well-known that there is a Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{2p/(2-p)}(\Omega)$ for $p \in [1, 2)$. If we look at the limiting Sobolev case $p = 2$, then $H_0^1(\Omega) := W_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \geq 1$, but $H_0^1(\Omega) \not\hookrightarrow L^\infty(\Omega)$. To fill this gap, it is natural to look for the maximal growth function $g : \mathbb{R} \to \mathbb{R}_+$ such that

$$\sup_{\substack{u \in H_0^1(\Omega) \\ \|\nabla u\|_2 \leq 1}} \int_\Omega g(u)dx < \infty,$$

where $\|\nabla u\|_2^2 = \int_\Omega |\nabla u|^2 dx$ denotes the Dirichlet norm of u. Pohozaev [20] and Trudinger [23] proved independently that the maximal growth is of ex-

[1]e-mail:mhashizume@hiroshima-u.ac.jp

ponential type and more precisely that there exists a constant α such that

$$\sup_{\substack{u \in H_0^1(\Omega) \\ \|\nabla u\|_2 \le 1}} \int_\Omega e^{\alpha u^2} dx < \infty.$$

Later, this inequality was sharpened by Moser [14] as follows:

$$\sup_{\substack{u \in H_0^1(\Omega) \\ \|\nabla u\|_2 \le 1}} \int_\Omega e^{\alpha u^2} dx \begin{cases} < C|\Omega| & \text{if} \quad \alpha \le 4\pi \\ = \infty & \text{if} \quad \alpha > 4\pi. \end{cases} \tag{1}$$

Lions [13] showed that for (1) there is a loss of compactness at the limiting exponent $\alpha = 4\pi$. However, despite the loss of compactness, the existence of a function which attains the supremum in (1) for $\alpha = 4\pi$ is shown by Carleson and Chang [2] if Ω is a unit ball. This result was extended to arbitrary bounded domains in $\mathbb{R}^2$ by Flucher [6].

In this paper, we study the properties of maximizers of the Trudinger-Moser functional

$$E_\alpha(u) := \int_\Omega \left(e^{\alpha u^2} - 1 \right) dx, \quad \alpha > 0$$

constrained to the manifold

$$\Sigma_\lambda := \left\{ u \in H^1(\Omega) \,\middle|\, \int_\Omega \left(|\nabla u|^2 + \lambda u^2 \right) dx = 1 \right\}$$

or

$$\Sigma_\lambda^0 := \left\{ u \in H_0^1(\Omega) \,\middle|\, \int_\Omega \left(|\nabla u|^2 + \lambda u^2 \right) dx = 1 \right\},$$

where $\lambda > 0$ is a parameter. By considering a transformation $u_\lambda(x) = u((x-p)/\sqrt{\lambda})$ for $u \in H^1(\Omega)$, $\lambda > 0$ and $p \in \mathbb{R}^2$, the existence of a maximizer for $\sup_{u \in \Sigma_\lambda} E_\alpha(u)$ on Ω is equivalent to that for $\sup_{u \in \Sigma_1} E_\alpha(u)$ on $\Omega_\lambda := \left\{ \sqrt{\lambda}x + p \,\middle|\, x \in \Omega \right\}$. The situation of Σ_λ^0 is same. By means of the parameter λ, we focus on asymptotic behavior of maximizers for the Trudinger-Moser inequalities on the scaling of Ω.

It is known that $\sup_{u \in \Sigma_\lambda} E_\alpha(u)$ is attained for $\alpha \in (0, 2\pi)$ and $\lambda > 0$ by the continuity of E_α with respect to weak convergence sequence in Σ_λ. In the critical case $\alpha = 2\pi$, by Yang [24], it is shown that $\sup_{u \in \Sigma_\lambda} E_{2\pi}(u)$ is attained for all $\lambda > 0$. Similarly, $\sup_{u \in \Sigma_\lambda^0} E_\alpha(u)$ is attained for $\alpha \in (0, 4\pi)$ and $\lambda > 0$, and it is proved that $\sup_{u \in \Sigma_\lambda^0} E_{4\pi}(u)$ is attained for $\lambda > 0$ by Ruf [21].

Asymptotic behaviors of critical points for $E_\alpha|_{\Sigma_\lambda}$ were considered in the subcritical case $\alpha \in (0, 2\pi)$ by the author [8]. In [8], the following Euler-Lagrange equation of critical points for $E_\alpha|_{\Sigma_\lambda}$ was studied.

$$\begin{cases} -\Delta u + \lambda u = \frac{u e^{\alpha u^2}}{\int_\Omega u^2 e^{\alpha u^2} dx} & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

In the case of large λ, it is shown that shape of maximizers of $\sup_{u \in \Sigma_\lambda} E_\alpha(u)$ depends on the exponent α. There exists $\alpha_* \in (0, 2\pi)$ such that for $\alpha \in (\alpha_*, 2\pi)$ any maximizer of $\sup_{u \in \Sigma_\lambda} E_\alpha(u)$ possesses a single spike-layer with its unique peak locating on the boundary of Ω. On the other hand, for $\alpha \in (0, \alpha_*)$ a limit of maximizers vanishes in the sense of $C(\overline{\Omega})$ as $\lambda \to \infty$. In the case of small λ, all positive critical points for $E_\alpha|_{\Sigma_\lambda}$ are close to $(\lambda|\Omega|)^{-1/2}$, which is the constant solution of (2). However, the critical case $\alpha = 2\pi$ was not dealt with in [8]. In this study, we consider the critical case $\alpha = 2\pi$, and then the asymptotic expansion of the best constant $\sup_{u \in \Sigma_\lambda} E_\alpha(u)$ for $\alpha \in (\alpha_*, 2\pi]$.

The first result we prove is the following.

Theorem 1.1. *Assume that u_λ is a maximizer of $\sup_{u \in \Sigma_\lambda} E_{2\pi}(u)$ for large λ. Then, there exist positive constants M_1 and M_2 independent of λ such that*

$$M_1 \le \sup_{x \in \Omega} u_\lambda(x) \le M_2$$

holds, and u_λ has a unique maximum which is attained at a point on $\partial\Omega$.

In addition to Theorem 1.1, we observe that u_λ is sufficiently small outside a small ball centered at the maximum point. Then, similar to the case of $\alpha \in (\alpha_*, 2\pi)$, maximizers for $\sup_{u \in \Sigma_\lambda} E_{2\pi}(u)$ exhibit the phenomenon of point condensation. The proof of the theorem is based on blow-up analysis and the techniques in [8].

To state the next result, we define a constant α_* introduced in [8]. This is defined by

$$\alpha_* := \inf \{\alpha \in (0, 2\pi) \mid I_\alpha > \alpha\},$$

where

$$I_\alpha := \sup_{\substack{u \in H^1(\mathbb{R}^2_+) \\ \int_{\mathbb{R}^2_+} \left(|\nabla u|^2 + u^2\right) dx \le 1}} \int_{\mathbb{R}^2_+} \left(e^{\alpha u^2} - 1\right) dx$$

and $\mathbb{R}^2_+ := \{x \in \mathbb{R}^2 \mid x_2 > 0\}$ is the half space. Note that $\alpha_* \in (0, 2\pi)$ holds and the constant α_* is the threshold in terms of existence of a maximizer of I_α, that is I_α is attained for $\alpha \in (\alpha_*, 2\pi]$ while I_α is not attained for $\alpha \in (0, \alpha_*)$ (see Appendix in [8]). The next result is the behavior of the peak of maximizer for $\sup_{u \in \Sigma_\lambda} E_\alpha(u)$.

Theorem 1.2. *Assume that $\alpha \in (\alpha_*, 2\pi]$, u_λ is a maximizer of $\sup_{u \in \Sigma_\lambda} E_\alpha(u)$ and $x_\lambda \in \partial\Omega$ satisfies $u_\lambda(x_\lambda) = \max_{x \in \overline{\Omega}} u_\lambda(x)$ for large λ. Then, we have*

$$\lim_{\lambda \to \infty} H(x_\lambda) = \max_{x \in \partial\Omega} H(x),$$

where $H(x)$ denotes curvature of $\partial\Omega$ at x.

In order to prove Theorem 1.2, we consider the asymptotic expansion of $\sup_{u \in \Sigma_\lambda} E_\alpha(u)$. Through translation and rotation of the coordinate system for a neighborhood N of x_λ, $\partial\Omega \cap N$ can be represented by

$$x_2 = \frac{1}{2} H(x_\lambda) x_1^2 + o(x_1^2)$$

with the curvature $H(x_\lambda)$ at $x_\lambda \in \partial\Omega$. By means of the representation, we derive that

$$E_\alpha(u_\lambda) = \frac{1}{\lambda}\left\{ I_\alpha + \tau H(x_\lambda)\frac{1}{\sqrt{\lambda}} + o\left(\sqrt{\lambda}^{-1}\right)\right\}$$

as $\lambda \to \infty$, where τ is a positive constant. This is the key estimate to prove Theorem 1.2.

Next, we consider the case of Σ^0_λ. For $\beta \in (0, 4\pi]$ we define d_β and β_* by

$$d_\beta := \sup_{\substack{u \in H^1(\mathbb{R}^2) \\ \int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2\right) dx \le 1}} \int_{\mathbb{R}^2} \left(e^{\beta u^2} - 1\right) dx$$

and

$$\beta_* := \inf\left\{\beta \in (0, 4\pi) \mid d_\beta > \beta\right\},$$

It holds that $d_\beta = 2 I_{\beta/2}$ and $\beta_* = 2\alpha_*$ (see Appendix in [8]). Then, we obtain the following results.

Theorem 1.3. *Assume that* $\alpha \in (0, 4\pi]$ *and* v_λ *is a maximizer of* $\sup_{u \in \Sigma_\lambda^0} E_\alpha(u)$ *for large* λ. *Then the following statements hold:*

(I) If $\alpha \in (\beta_*, 4\pi]$, *then there exist positive constants* Λ_1, M_1 *and* M_2 *such that for any* $\lambda > \Lambda_1$ *we have*

$$M_1 \leq \sup_{x \in \Omega} v_\lambda(x) \leq M_2.$$

(II) If $\alpha \in (0, \beta_*)$, *then we have*

$$v_\lambda \to 0 \quad \text{in} \quad C^0(\overline{\Omega})$$

and

$$\int_\Omega |\nabla v_\lambda|^2 dx \to 0, \quad \lambda \int_\Omega v_\lambda^2 dx \to 1$$

as $\lambda \to \infty$.

Theorem 1.4. *Assume that* $\alpha \in (\beta_*, 4\pi]$, v_λ *is a maximizer of* $\sup_{u \in \Sigma_\lambda^0} E_\alpha(u)$ *and* $x_\lambda \in \overline{\Omega}$ *satisfies* $v_\lambda(x_\lambda) = \max_{x \in \overline{\Omega}} v_\lambda(x)$ *for large* λ. *Then, we have*

$$\lim_{\lambda \to \infty} \operatorname{dist}(x_\lambda, \partial\Omega) = \max_{x \in \Omega} \operatorname{dist}(x, \partial\Omega).$$

In the case of Σ_λ^0, maximizers v_λ exhibits point condensation for $\alpha \in (\beta_*, 4\pi]$ and vanishing phenomenon for $\alpha \in (0, \beta_*)$. The asymptotic expansion of $\sup_{u \in \Sigma_\lambda^0} E_\alpha(u)$ for $\alpha \in (\beta_*, 4\pi]$ is

$$\sup_{u \in \Sigma_\lambda^0} E_\alpha(u) = \frac{1}{\lambda} \left\{ d_\alpha + \exp\left[-\gamma\sqrt{\lambda} \operatorname{dist}(x_\lambda, \partial\Omega) + o(\sqrt{\lambda}) \right] \right\}$$

as $\lambda \to \infty$, where γ is a positive constant. The expansion leads Theorem 1.4.

Concerning asymptotic behavior of least energy solutions for semilinear elliptic equations, in [12, 18, 16], they considered the following Neumann problem for power type nonlinearity:

$$\begin{cases} -\varepsilon^2 \Delta u + u = f(u) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

where ε is a parameter and f satisfies some conditions with $f(t) = O(t^p)$ as $t \to \infty$ for $p > 1$. The following Dirichlet boundary condition is also considered in [19].

$$\begin{cases} -\varepsilon^2 \Delta u + u = f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

In the case of small ε, it is proved by [12, 18, 16] that a solution at this least energy level for the Neuamnn problem (3) possesses just one local maximum point, which lies on the boundary, and concentrates (up to subsequences) around a point where mean curvature maximizes. On the other hand, Ni and Wei [19] show that a least energy solution of the Dirichlet problem (4) necessarily concentrates around a "most centered point" of the domain, namely around a point of maximum distance to the boundary. In both problems the method employed consists of a combination of the variational characterization of the solutions and exact estimates of the value of the energy functional based on a precise asymptotic analysis of the solutions.

We remark that if $f(u) = u^p$ in (3) or (4), then least energy solutions attain the best constant of corresponding minimization problem

$$S_N := \inf_{\substack{u \in H^1(\Omega) \\ u \not\equiv 0}} \frac{\int_\Omega (\varepsilon |\nabla u|^2 + u^2) dx}{\left(\int_\Omega |u|^{p+1} dx \right)^{2/(p+1)}} \quad \text{or} \quad S_D := \inf_{\substack{u \in H^1_0(\Omega) \\ u \not\equiv 0}} \frac{\int_\Omega (\varepsilon |\nabla u|^2 + u^2) dx}{\left(\int_\Omega |u|^{p+1} dx \right)^{2/(p+1)}},$$

and the opposite statement is also true provided suitable normalization. However, the relationship between least energy solution of an equation and extremal function for corresponding variational problem is open for general setting on f including exponential nonlinearity. Moreover, the Euler-Lagrange equation of maximizers for the Trudinger-Moser inequalities is nonlocal equation. Although there is the difference, in this paper, we apply the methods of [12, 18, 16, 19] to the framework of maximizers for the Trudinger-Moser equation.

This paper is organized as follows. In Section 2, we will prove Theorems 1.1 and 1.2. In Section 3, we will prove Theorems 1.3 and 1.4. To prove Theorems 1.1 and 1.3, we use the blow-up analysis and the strategy in [8]. The proof of Theorems 1.2 and 1.4 follows the techniques in [4].

2. Maximizer for $\sup_{u\in\Sigma_\lambda} E_\alpha(u)$: Proofs of Theorems 1.1 and 1.2

2.1. Proof of Theorem1.1

In this section we prove Theorem 1.1. We study a nonlocal elliptic equation to derive the asymptotic behavior of u_λ.

Assume that λ_n is a sequence with $\lambda_n \to \infty$ as $n \to \infty$ and $u_n := u_{\lambda_n}$ is a maximizer for $\sup_{u\in\Sigma_{\lambda_n}} E_{2\pi}(u)$. First, we prove the existence of a constant C such that

$$\sup_{x\in\Omega} u_\lambda(x) \le C,$$

where C is independent of n. For simplicity, we write $c_n = \sup_{x\in\Omega} u_n(x)$. Assume the contrary that $c_n \to \infty$ as $n \to \infty$ and derive a contradiction. To derive a contradiction, we estimate the value of $E_{2\pi}(u_n)$. We will prove the lower bound

$$I_{2\pi} \le \liminf_{n\to\infty} \lambda_n E_{2\pi}(u_n).$$

On the other hand, we will derive the upper bound

$$\limsup_{n\to\infty} \lambda_n E_{2\pi}(u_n) \le \frac{\pi}{2} e^{4\pi K+1},$$

where K is an explicit constant. Then, it is known that $\pi e^{4\pi K+1} < d_{4\pi}$ by [11]. Combining these results and the fact that $d_{4\pi} = 2I_{2\pi}$, we derive a contradiction.

Proposition 2.1. *Assume that u_λ is a maximizer for $\sup_{u\in\Sigma_\lambda} E_{2\pi}(u)$ with large λ. Then, we have*

$$I_{2\pi} \le \liminf_{\lambda\to\infty} \lambda E_{2\pi}(u_\lambda).$$

Proof. Without loss of generality, we may assume that $0 \in \partial\Omega$ and $\Omega \subset \mathbb{R}^2_+$. Let $U \in H^1(\mathbb{R}^2_+)$ be a maximizer of $I_{2\pi}$ and set

$$U_n(x) := U\left(\sqrt{\lambda_n} x\right).$$

Since $\int_{\mathbb{R}^2_+} (|\nabla U|^2 + U^2)\, dx = 1$, we have

$$\int_\Omega \left(|\nabla U_n|^2 + \lambda_n U_n^2\right) dx \le \int_{\mathbb{R}^2_+} \left(|\nabla U_n|^2 + \lambda_n U_n^2\right) dx = \int_{\mathbb{R}^2_+} \left(|\nabla U|^2 + U^2\right) dx = 1.$$

Then, it follows that

$$\begin{aligned}
E_{2\pi}(u_n) &\geq \int_{\Omega}\left(e^{2\pi U_n^2}-1\right)dx \\
&\geq \int_{\Omega\cap B_{R/\sqrt{\lambda_n}}}\left(e^{2\pi U_n^2}-1\right)dx \\
&= \lambda_n^{-1}\int_{\Omega_{\lambda_n}\cap B_R}\left(e^{2\pi U^2}-1\right)dx,
\end{aligned}$$

where $\Omega_{\lambda_n} := \left\{\sqrt{\lambda_n}x \;\middle|\; x\in\Omega\right\}$. The smoothness of the boundary of Ω gives

$$\liminf_{n\to\infty}\lambda_n E_{2\pi}(u_n) \geq \int_{B_R\cap\mathbb{R}^2_+}\left(e^{2\pi U^2}-1\right)dx.$$

By letting $R\to\infty$, we conclude that

$$\liminf_{n\to\infty}\lambda_n E_{2\pi}(u_n) \geq I_{2\pi}.$$

□

A maximizer u_n satisfies the following Euler-Lagrange equation.

$$\begin{cases} -\Delta u_n + \lambda_n u_n = L_n u_n e^{2\pi u_n^2} & \text{in} \quad \Omega, \\ \frac{\partial u_n}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5}$$

where L_n is the Lagrange multiplier characterized by $\left(\int_\Omega u_n^2 e^{2\pi u_n^2}dx\right)^{-1}$. A maximum point of u_n is denoted by x_n. In the following, we assume the contrary that $c_n = \sup_{x\in\Omega} u_n(x)\to\infty$ as $n\to\infty$.

Here, we introduce a diffeomorphism straightening a boundary portion around a point on $\partial\Omega$, which was introduced in [12, 18, 16]. Fix $P\in\partial\Omega$. Through translation and rotation of the coordinate system we may assume that P is the origin and the inner normal to $\partial\Omega$ at P is pointing in the direction of the positive x_2-axis. In a neighborhood N of P, $\partial\Omega\cap N$ can be represented by

$$x_2 = \psi(x_1) = \frac{1}{2}H(P)x_1^2 + o(x_1^2),$$

where H is the curvature of $\partial\Omega$ at P. Define a map $x = \Phi(y) = (\Phi_1(y), \Phi_2(y))$ by

$$\Phi_1(y) = y_1 - y_2\frac{\partial\psi}{\partial x_1}(y_1), \quad \Phi_2(y) = y_2 + \psi(y_1). \tag{6}$$

Since $\psi'(0) = 0$, the differential map $D\Phi$ of Φ satisfies $D\Phi(0) = I$, the identity map. Thus, Φ has the inverse mapping $y = \Phi^{-1}(x)$ for small $|x|$. We write $\Psi(x) = (\Psi_1(x), \Psi_2(x))$ instead of $\Phi^{-1}(x)$.

We define r_n such that

$$r_n^{-2} = L_n c_n^2 e^{2\pi c_n^2}. \tag{7}$$

By the characterization of L_n, we see that

$$r_n^{-2} = \frac{c_n^2 e^{2\pi c_n^2}}{\int_\Omega u_n^2 e^{2\pi u_n^2} dx} \geq \lambda_n c_n^2$$

or

$$r_n \leq \left(\sqrt{\lambda_n} c_n\right)^{-1}. \tag{8}$$

Then, we derive the following results.

Lemma 2.2. *We have*

$$\operatorname{dist}(x_n, \partial\Omega) = o(r_n)$$

and

$$\lim_{R\to\infty} \lim_{n\to\infty} L_n \int_{\Omega\cap\Phi(B_{Rr_n}(P_n))} u_n^2 e^{2\pi u_n^2} dx = 1, \tag{9}$$

where $P_n = \Psi(x_n)$.

Proof. First, we prove that $\operatorname{dist}(x_n, \partial\Omega) = O(r_n)$. If $\operatorname{dist}(x_n, \partial\Omega)/r_n \to \infty$, we define $\Omega_n := \{(x - x_n)/r_n \mid x \in \Omega\}$ and

$$\begin{cases} \phi_n(y) := c_n^{-1} u_n(r_n y + x_n) & y \in \Omega_n, \\ \eta_n(y) := c_n(u_n(r_n y + x_n) - c_n) & y \in \Omega_n. \end{cases}$$

Then, ϕ_n and η_n satisfy

$$-\Delta_y \phi_n + \lambda_n r_n^2 \phi_n = c_n^{-2} \phi_n e^{\alpha c_n^2 \left(\phi_n^2 - 1\right)},$$
$$-\Delta_y \eta_n + \lambda_n r_n^2 c_n^2 \phi_n = \phi_n e^{\alpha(1+\phi_n)\eta_n}. \tag{10}$$

Since (8) and $\operatorname{dist}(x_n, \partial\Omega)/r_n \to \infty$ hold, for any $R > 0$ there exists N such that $B_R(x_n) \subset \Omega_n$ for any $n \geq N$. Thus, by the elliptic regularity theory and the maximum principle, we see that

$$\phi_n \to \phi_0 \equiv 1 \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2), \quad -\Delta\phi_0 = 0 \quad \text{in} \quad \mathbb{R}^2.$$

Using the behavior of ϕ_n, we estimate $\lambda_n r_n^2 c_n^2$ in (10). Since $u_n \in \Sigma_{\lambda_n}$, we have

$$\begin{aligned} 1 &\geq \lambda_n \int_\Omega u_n^2 dx \geq \lambda_n c_n^2 \int_{B_{Rr_n}(x_n)} \left(\frac{u_n}{c_n}\right)^2 dx = \lambda_n c_n^2 r_n^2 \int_{B_R} \phi_n^2 dy \\ &= \lambda_n c_n^2 r_n^2 \int_{B_R} (1+o(1))^2 dy = \lambda_n c_n^2 r_n^2 |B_R|(1+o(1)) \end{aligned}$$

for any $R > 0$, and thus $\lambda_n c_n^2 r_n^2 \to 0$ as $n \to \infty$. Applying the elliptic regularity theory to (10), we have

$$\eta_n \to \eta_0 \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2), \quad -\Delta \eta_0 = e^{4\pi\eta_0} \quad \text{in} \quad \mathbb{R}^2.$$

Moreover, it follows that

$$\begin{aligned} \int_{\mathbb{R}^2} e^{4\pi\eta_0} dy &= \lim_{R\to\infty} \lim_{n\to\infty} \int_{B_R} \phi_n^2 e^{2\pi(1+\phi_n)\eta_n} dy \\ &\leq \lim_{R\to\infty} \lim_{n\to\infty} L_n \int_{B_{Rr_n}(x_n)} u_n^2 e^{2\pi u_n^2} dx \\ &\leq 1, \end{aligned} \tag{11}$$

and then, by the characterization result of [3], we have

$$\eta_0 = -\frac{1}{2\pi} \log\left(1 + \frac{\pi}{2}|y|^2\right).$$

On the other hand, by a direct computation, we have

$$\int_{\mathbb{R}^2} e^{4\pi\eta_0} dy = 2,$$

which contradicts (11). Hence $\operatorname{dist}(x_n, \partial\Omega) = O(r_n)$.

Next, we prove $\operatorname{dist}(x_n, \partial\Omega) = o(r_n)$. One may assume that $x_n \to x_0 \in \partial\Omega$ by passing to a subsequence if necessary. Consider the diffeomorphism $y = \Psi(x)$ that straightens a boundary portion near x_0, as in (6). We may assume that $\Phi = \Psi^{-1}$ is defined in an open set containing the closed ball $\overline{B_{2\kappa}}, \kappa > 0$, and that $P_n := \Psi(x_n) \in B^+_\kappa$ for all n. Put

$$\tilde{u}_n(y) := u_n(\Phi(y)) \quad \text{for} \quad y \in \overline{B^+_{2\kappa}}$$

and extend it to $\overline{B_{2\kappa}}$ by reflection:

$$\overline{u}_n(y) := \begin{cases} \tilde{u}_n(y) & \text{if} \quad y \in \overline{B^+_{2\kappa}}, \\ \tilde{u}_n((y_1, -y_2)) & \text{if} \quad y \in B^-_{2\kappa}, \end{cases}$$

where $B^-_{2\kappa} := \left\{ y \in \overline{B_{2\kappa}} \;\middle|\; y_2 < 0 \right\}$. Moreover, we define a scaled function $\hat{u}_n(z)$ by

$$\hat{u}_n(z) := \overline{u}_n(r_n z + P_n) \quad \text{for} \quad z \in \overline{B_{\kappa/r_n}},$$

and then ϕ_n and η_n are defined by

$$\phi_n(z) := c_n^{-1} \hat{u}_n(z),$$

$$\eta_n(z) := c_n(\hat{u}_n(z) - c_n).$$

Let $P_n := (p_n, q_n r_n)$. The condition $\operatorname{dist}(x_n, \partial\Omega) = O(r_n)$ implies that $q_n < \infty$. By (5), ϕ_n and η_n satisfy the following elliptic equations:

$$-\sum_{i,j=1}^{2} a^n_{ij}(z) \frac{\partial^2 \phi_n}{\partial z_i \partial z_j} - r_n \sum_{j=1}^{2} b^n_j(z) \frac{\partial \phi_n}{\partial z_j} + \lambda_n r_n^2 \phi_n = c_n^{-2} \phi_n e^{2\pi c_n^2 \left(\phi_n^2 - 1\right)},$$

$$-\sum_{i,j=1}^{2} a^n_{ij}(z) \frac{\partial^2 \eta_n}{\partial z_i \partial z_j} - r_n \sum_{j=1}^{2} b^n_j(z) \frac{\partial \eta_n}{\partial z_j} + \lambda_n r_n^2 c_n^2 \phi_n = \phi_n e^{2\pi(1+\phi_n)\eta_n},$$

where a^n_{ij}, b^n_j are defined as follows: First, put

$$\begin{aligned} a_{ij}(y) &= \sum_{\ell=1}^{2} \frac{\partial \Psi_i}{\partial x_\ell}(\Phi(y)) \frac{\partial \Psi_j}{\partial x_\ell}(\Phi(y)) \quad 1 \le i, j \le 2 \\ b_j(y) &= (\Delta \Psi_j)(\Phi(y)) \quad 1 \le j \le 2. \end{aligned}$$

Then, set

$$a^n_{ij}(z) = \begin{cases} a_{ij}(P_n + r_n z) & z_2 \ge -q_n, \\ (-1)^{\delta_{i2} + \delta_{j2}} a_{ij}((p_n + r_n z_1, -(q_n + z_2) r_n) & z_2 < q_n, \end{cases}$$

$$b^n_j(z) = \begin{cases} b_j(P_n + r_n z) & z_2 \ge -q_n, \\ (-1)^{\delta_{j2}} b_j((p_n + r_n z_1), -(q_n + z_2) r_n) & z_2 < -q_n, \end{cases}$$

where δ_{ij} is the Kronecker symbol. Using the elliptic regularity theory, we have

$$\phi_n \to \phi_0 \equiv 1 \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2), \quad -\Delta\phi_0 = 0 \quad \text{in} \quad \mathbb{R}^2,$$
$$\eta_n \to \eta_0 \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2), \quad -\Delta\eta_0 = e^{4\pi\eta_0} \quad \text{in} \quad \mathbb{R}^2.$$

Computing $\int_{\mathbb{R}^2} e^{4\pi\eta_0}dz$ in the same manner as in (11), we have

$$\int_{\mathbb{R}^2} e^{4\pi\eta_0}dz \le \lim_{R\to\infty}\lim_{n\to\infty} 2L_n \int_{\Omega\cap\Phi(B_{Rr_n}(P_n))} u_n^2 e^{2\pi u_n^2}dx \le 2. \tag{12}$$

Hence, we see that

$$\eta_0 = -\frac{1}{2\pi}\log\left(1+\frac{\pi}{2}|z|^2\right),$$

and then, $q_n \to 0$, which implies that $\operatorname{dist}(x_n, \partial\Omega) = o(r_n)$.

By a direct computation, we have

$$\int_{\mathbb{R}^2} e^{4\pi\eta_0}dz = 2.$$

The above computation and (12) yield (9). □

By Lemma 2.2, we may assume that, up to a subsequence, $x_n \to x_0 \in \partial\Omega$. For $A > 1$, let $u_n^A = \min\{u_n, c_n/A\}$. We have the following result.

Lemma 2.3. *For any $A > 1$, we have*

$$\limsup_{n\to\infty} \int_\Omega \left(|\nabla u_n^A|^2 + \lambda_n |u_n^A|^2\right) dx \le \frac{1}{A}.$$

Proof. Multiplying (5) by u_n^A, integrating over Ω and using (9), we have

$$\begin{aligned}
&\int_\Omega \left(\nabla u_n \nabla u_n^A + \lambda_n u_n u_n^A\right) dx \\
\le\ & L_n \int_{\Omega\cap\Phi(B_{Rr_n}(P_n))} u_n u_n^A e^{2\pi u_n^2}dx + L_n \int_{\Omega\setminus\Phi(B_{Rr_n}(P_n))} u_n^2 e^{2\pi u_n^2}dx \\
=\ & \frac{1}{A} + o_n(1) + o_R(1),
\end{aligned}$$

where $o_n(1) \to 0$ as $n \to \infty$ and $o_R(1) \to 0$ as $R \to \infty$. Since

$$\int_\Omega \left(|\nabla u_n^A|^2 + \lambda_n |u_n^A|^2\right) dx \le \int_\Omega \left(\nabla u_n \nabla u_n^A + \lambda_n u_n u_n^A\right) dx,$$

we deduce that

$$\int_\Omega \left(|\nabla u_n^A|^2 + \lambda_n |u_n^A|^2\right) dx \le \frac{1}{A} + o_n(1) + o_R(1).$$

Letting $R \to \infty$ after $n \to \infty$, we derive Lemma 2.3. □

Lemma 2.4. *There exists a positive constant C such that*

$$\liminf_{n\to\infty} \frac{\lambda_n}{c_n^2 L_n} \ge C$$

holds.

For the proof the lemma, we recall the following result.

Proposition 2.5. *There exist a operator T and a positive constant M such that*

$$T : H^1(\Omega) \to H^1(\mathbb{R}^2)$$

and

$$\int_{\mathbb{R}^2} \left(|\nabla(Tu)|^2 dx + |Tu|^2\right) dx \le M \int_\Omega \left(|\nabla u|^2 + u^2\right) dx, \tag{13}$$

where M is independent of the scaling of Ω.

Proof. We have

$$\begin{aligned} \lambda_n E_{2\pi}(u_n) &= \lambda_n \int_{\left[u_n > \frac{c_n}{A}\right]} \left(e^{2\pi u_n^2} - 1\right) dx + \lambda_n \int_{\left[u_n \le \frac{c_n}{A}\right]} \left(e^{2\pi u_n^2} - 1\right) dx \\ &\le A^2 \frac{\lambda_n}{c_n^2 L_n} + \lambda_n \int_\Omega \left(e^{2\pi |u_n^A|^2} - 1\right) dx. \end{aligned} \tag{14}$$

Using (13) and Lemma 2.3, we have

$$\lambda_n \int_\Omega \left(e^{2\pi|u_n^A|^2} - 1\right) dx \le \int_{\mathbb{R}^2} \left(e^{2\pi |Tu_n^A(x/\sqrt{\lambda_n})|^2} - 1\right) dx \le d_{2\pi} \tag{15}$$

for large A. Moreover, by Proposition 2.1, the convexity of the function $e^s - 1$ and the existence of maximizer for $I_{2\pi}$, we see that

$$\liminf_{n\to\infty} E_{2\pi}(u_n) \ge I_{2\pi} > 2I_\pi = d_{2\pi}. \tag{16}$$

Combining (14)-(16), we have

$$\delta \le A^2 \liminf_{n\to\infty} \frac{\lambda_n}{c_n^2 L_n}$$

for some positive constant δ. Hence, we conclude Lemma 2.4. □

Set a point $x_n^* \in \partial\Omega$ such that $|x_n - x_n^*| = \mathrm{dist}(x_n, \partial\Omega)$. In the following, we consider $\hat{u}_n(x) = u_n(x/\sqrt{\lambda_n} + x_n^*)$ and the equation

$$\begin{cases} -\Delta \hat{u}_n + \hat{u}_n = \frac{L_n}{\lambda_n}\hat{u}_n e^{2\pi \hat{u}_n^2} & \text{in} \quad \Omega_n, \\ \frac{\partial \hat{u}_n}{\partial \nu} = 0 & \text{on} \quad \partial\Omega_n, \end{cases} \tag{17}$$

where $\Omega_n := \left\{ \sqrt{\lambda_n}(x - x_n^*) \;\middle|\; x \in \Omega \right\}$. Obviously, $\sup_{x \in \Omega_n} \hat{u}_n = c_n$. Define $\hat{x}_n$ by a maximum point of $\hat{u}_n$ and put $\hat{r}_n = \sqrt{\lambda_n} r_n$, where r_n is defined in (7). By Lemma 2.2, we observe that

$$|\hat{x}_n| = \mathrm{dist}(\hat{x}_n, \partial\Omega_n) = o(\hat{r}_n) \tag{18}$$

and that

$$\lim_{R\to\infty} \lim_{n\to\infty} \frac{L_n}{\lambda_n} \int_{\Omega_n \cap \Phi\left(B_{R\hat{r}_n}(\hat{P}_n)\right)} \hat{u}_n^2 e^{2\pi \hat{u}_n^2} dx = 1, \tag{19}$$

where $\hat{P}_n = \Psi(\hat{x}_n)$. We also have

$$\int_{\Omega_n} \left(|\nabla \hat{u}_n^A|^2 + |\hat{u}_n^A|^2 \right) dx \le \frac{1}{A}$$

for any $n \in \mathbb{N}$, $A > 1$ and $\hat{u}_n^A = \min\{\hat{u}_n, c_n/A\}$ by Lemma 2.3.

Lemma 2.6. *For any $\psi \in C(\mathbb{R}^2)$ with $\|\psi\|_{L^\infty(\mathbb{R}^2)} < \infty$ it follows that*

$$\lim_{n\to\infty} \frac{L_n}{\lambda_n} \int_{\Omega_n} \psi c_n \hat{u}_n e^{2\pi \hat{u}_n^2} dx = \psi(0).$$

Proof. Fix $\psi \in C(\mathbb{R}^2)$. We divide $L_n \lambda_n^{-1} \int_{\Omega_n} \psi c_n \hat{u}_n e^{2\pi \hat{u}_n^2} dx$ into three parts as follows.

$$\begin{aligned} \frac{L_n}{\lambda_n} \int_{\Omega_n} \psi c_n \hat{u}_n e^{2\pi u_n^2} dx &= \frac{L_n}{\lambda_n} \int_{\Omega_n \cap \Phi\left(B_{R\hat{r}_n}(\hat{P}_n)\right)} \psi c_n \hat{u}_n e^{2\pi u_n^2} dx \\ &\quad + \frac{L_n}{\lambda_n} \int_{\left[\Omega_n \setminus \overline{\Phi\left(B_{R\hat{r}_n}(\hat{P}_n)\right)}\right] \cap \left[\hat{u}_n > \frac{c_n}{A}\right]} \psi c_n \hat{u}_n e^{2\pi u_n^2} dx \\ &\quad + \frac{L_n}{\lambda_n} \int_{\left[\Omega_n \setminus \overline{\Phi\left(B_{R\hat{r}_n}(\hat{P}_n)\right)}\right] \cap \left[\hat{u}_n \le \frac{c_n}{A}\right]} \psi c_n \hat{u}_n e^{2\pi \hat{u}_n^2} dx \\ &= I_1 + I_2 + I_3. \end{aligned}$$

For I_1, by (18) and (19), setting $\eta_0 = -(2\pi)^{-1}\log(1+\pi|z|^2/2)$we have

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^2_+\cap B_R} \psi\left(\Phi(\hat{r}_n z + \hat{P}_n)\right)(1+o(1))e^{2\pi(2+o(1))(\eta_0+o(1))}dz \\ &= (\psi(0)+o_n(1))\left(1+o_n(1)+o_R(1)\right), \end{aligned}$$

where $o_n(1)\to 0$ as $n\to\infty$ for each R and $o_R(1)\to 0$ as $R\to\infty$. Thus, letting $n\to\infty$ first, and then $R\to\infty$, we derive that

$$\lim_{n\to\infty} I_1 = \psi(0).$$

For I_2, it follows that

$$|I_2| \le \|\psi\|_{L^\infty(\mathbb{R}^2)} \frac{AL_n}{\lambda_n}\int_{\Omega_n\setminus\overline{\Phi(B_{R\hat{r}_n}(P_n))}} \hat{u}_n^2 e^{2\pi\hat{u}_n^2}dx$$

By (19) and the fact that $L_n\lambda_n^{-1}\int_{\Omega_n}\hat{u}_n^2 e^{2\pi\hat{u}_n^2}dx = 1$, we deduce that

$$\limsup_{n\to\infty}|I_2| = 0.$$

Finally, we estimate I_3. It holds that

$$\begin{aligned} &|I_3| \\ \le\ & \|\psi\|_{L^\infty(\mathbb{R}^2)}\frac{c_nL_n}{\lambda_n}\int_{\Omega_n}\hat{u}_n^A e^{2\pi|\hat{u}_n^A|^2}dx \\ =\ & \|\psi\|_{L^\infty(\mathbb{R}^2)}\frac{c_nL_n}{\lambda_n}\left[\int_{\Omega_n}\hat{u}_n^A\left(e^{2\pi|\hat{u}_n^A|^2}-1\right)dx + \int_{\Omega_n}\hat{u}_n^A dx\right] \\ \le\ & \|\psi\|_{L^\infty(\mathbb{R}^2)}\frac{c_nL_n}{\lambda_n}\left[\int_{\Omega_n}\hat{u}_n^A\left(e^{2\pi|\hat{u}_n^A|^2}-1\right)dx + \frac{L_n}{\lambda_n}\int_{\Omega_n}\hat{u}_n e^{2\pi\hat{u}_n^2}dx\right] \\ \le\ & \|\psi\|_{L^\infty(\mathbb{R}^2)}\frac{c_nL_n}{\lambda_n}\left(\int_{\Omega_n}|\hat{u}_n^A|^2dx\right)^{\frac{1}{2}}\left[\int_{\Omega_n}\left(e^{4\pi|\hat{u}_n^A|^2}-1\right)dx\right]^{\frac{1}{2}} \\ & +\|\psi\|_{L^\infty(\mathbb{R}^2)}\frac{c_nL_n}{\lambda_n}\left(\frac{L_n}{\lambda_n}\int_{\Omega_n}\hat{u}_n e^{2\pi u_n^2}dx\right) \\ \le\ & \|\psi\|_{L^\infty(\mathbb{R}^2)}\frac{c_nL_n}{\lambda_n}\left(\frac{d_{4\pi}}{\sqrt{A}} + \frac{L_n}{\lambda_n}\int_{\Omega_n}\hat{u}_n e^{2\pi\hat{u}_n^2}dx\right) \end{aligned}$$

provided that A satisfies $M \le A$, where M is a constant as in (13). By Lemma 2.4, we have $c_nL_n/\lambda_n = o(1)$ and $L_n/\lambda_n = o(1)$. Hence, we derive that

$$\frac{L_n}{\lambda_n}\int_{\Omega_n}\psi c_n\hat{u}_n e^{2\pi\hat{u}_n^2}dx=\psi(0)+o\left(\frac{L_n}{\lambda_n}\int_{\Omega_n}c_n\hat{u}_n e^{2\pi\hat{u}_n^2}dx\right)+o(1)$$

for any $\psi\in C(\mathbb{R}^2)$ with $\|\psi\|_{L^\infty(\mathbb{R}^2)}<\infty$. Consequently, Lemma 2.6 holds for $\psi\equiv 1$ first, and then Lemma 2.6 holds for any $\psi\in C(\mathbb{R}^2)$ satisfying $\|\psi\|_{L^\infty(\mathbb{R}^2)}<\infty$. □

Lemma 2.7. *We have*

$$\limsup_{n\to\infty}\int_{\Omega_n\setminus B_R}\left(|\nabla\hat{u}_n|^2+\hat{u}_n^2\right)dx=O(R^{-1})$$

as $R\to\infty$.

Proof. Consider a function $\tau\in C^\infty(\mathbb{R}^2)$ such that

$$\tau(x)=\begin{cases}0 & \text{if}\quad x\in B_{R_0},\\ 1 & \text{if}\quad x\in\mathbb{R}^2\setminus B_{2R_0},\end{cases}\qquad |\nabla\tau(x)|\le\frac{2}{R_0}.$$

Then, multiplying (17) by $\tau\hat{u}_n$ and integrating on Ω_n, we have

$$\int_{\Omega_n}\tau\left(|\nabla\hat{u}_n|^2+\hat{u}_n^2\right)dx+\int_{\Omega_n}u_n\nabla\tau\nabla u_n dx=\frac{L_n}{\lambda_n}\int_{\Omega_n}\tau\hat{u}_n^2e^{2\pi\hat{u}_n^2}dx.$$

Using Lemma 2.6, we derive that

$$\begin{aligned}&\int_{\Omega_n\setminus B_{2R_0}}\left(|\nabla\hat{u}_n|^2+\hat{u}_n^2\right)dx\\ \le\ &\|\nabla\tau\|_{L^\infty(\mathbb{R}^2)}\left(\int_{\Omega_n}|\nabla\hat{u}_n|^2dx\right)^{\frac{1}{2}}\left(\int_{\Omega_n}\hat{u}_n^2dx\right)^{\frac{1}{2}}+o(1)\\ \le\ &\frac{2}{R_0}+o(1).\end{aligned}$$

Hence, we obtain desired estimate. □

Lemma 2.8. *There exist* n_0*,* $R_0>0$ *and* $C_0>0$ *such that for any* $n\ge n_0$ *we have*

$$\sup_{x\in\Omega_n\setminus B_{2R_0}(\hat{x}_n)}c_n\hat{u}_n(x)\le C_0.$$

Proof. For the proof, we employ Proposition 9.20 in [7] as follows.

Proposition 2.9. *Let $u \in W^{2,2}(D)$ and L is an elliptic operator. Suppose that $Lu \geq f$, where $f \in L^2(D)$. Then, for any ball $B_{2R}(y) \subset D$ and $p > 0$, we have*

$$\sup_{x \in B_R(y)} u(x) \leq C \left\{ \left(\frac{1}{|B_{2R}|} \int_{B_{2R}} (u^+)^p \right)^{\frac{1}{p}} + \frac{R}{\Lambda_1} \|f\|_{L^2(B_{2R})} \right\},$$

where $|B_{2R}|$ is the Lebesgue measure of B_{2R}, the constant Λ_1 denotes the minimum eigenvalue of the coefficient matrix of operator L and C is independent of D.

We apply the proposition to $\hat{u}_n$. By Lemma 2.7, for sufficiently large R_0 and a ball $B_{2\kappa}(y) \subset \Omega_n \setminus B_{2R_0}$ it follows that

$$\begin{aligned} & \int_{B_{2\kappa}(y)} \left(\frac{L_n}{\lambda_n} c_n \hat{u}_n e^{2\pi \hat{u}_n^2} \right)^2 dx \\ \leq & \frac{c_n^2 L_n^2}{\lambda_n^2} \left(\int_{B_{2\kappa}(y)} \hat{u}_n^4 dx \right)^{\frac{1}{2}} \left(\int_{B_{2\kappa}(y)} e^{8\pi \hat{u}_n^2} dx \right)^{\frac{1}{2}} \\ = & o(1). \end{aligned} \tag{20}$$

Moreover, we have

$$\int_{\Omega_n \setminus B_{2R_0}} c_n \hat{u}_n dx \leq \frac{L_n}{\lambda_n} \int_{\Omega_n} c_n \hat{u}_n e^{2\pi \hat{u}_n^2} dx \leq 1 + o(1).$$

Thus, by the estimate, (20) and Proposition 2.9 with $L = \Delta - 1$, $f = -L_n c_n \hat{u}_n e^{2\pi \hat{u}_n^2} / \lambda_n$ and $p = 1$, we have

$$\sup_{x \in B_\kappa(y)} c_n \hat{u}_n \leq \frac{C\kappa}{A}$$

for $B_{2\kappa}(y) \subset \Omega_n \setminus B_{2R_0}$. In the neighborhood around $\partial (\Omega_n \setminus B_{2R_0})$, defining $\hat{w}_n$ as the extension of $\hat{u}_n$ by the diffeomorphism straightening a boundary portion at each point of $\partial\Omega$ as in (6) and the reflection, we apply Proposition 2.9 to $\hat{w}_n$. Hence, Lemma 2.8 holds. □

Lemma 2.10. *Let R be sufficiently large. Then, there exists a positive constant C such that for any n and any $x \in \Omega_n \cap B_R \setminus \{\hat{x}_n\}$ we have*

$$c_n \hat{u}_n(x) \leq C \log \left(\frac{C}{|x - \hat{x}_n|} \right).$$

Proof. First, we recall properties of a function G_y which is a solution of

$$-\Delta G_y + G_y = \delta_y \quad \text{in} \quad \mathbb{R}^2.$$

By the characterization of G_y, the function is radially symmetric with respect to $y \in \mathbb{R}^2$, $G_y \in C^2_{loc}(\mathbb{R}^2 \setminus \{y\})$ and

$$\lim_{x \to y} \left[G_y(x) - \frac{1}{2\pi} \log \left(\frac{1}{|x-y|} \right) \right] = K.$$

with some positive constant K.

Fix $R > 0$ sufficiently large and $y \in \Omega_n \cap B_R$. Then, by the properties of G_y, the diffeomorphism straightening a boundary portion around $0 \in \partial\Omega$ as in (6) and the reflection, the solution of

$$\begin{cases} -\Delta h_y + h_y = 0 & \text{in} \quad \Omega_n \cap B_{2R}, \\ \frac{\partial h_y}{\partial \nu} = -\frac{\partial G_y}{\partial \nu} & \text{on} \quad \partial\Omega_n \cap B_{2R}, \\ h_y = -G_y & \text{on} \quad \Omega_n \cap \partial B_{2R} \end{cases}$$

satisfies

$$h_n(x) \le \frac{1}{2\pi} \log \left(\frac{C}{|x-y|} \right)$$

for any $x \in \Omega_n \cap B_{2R}$, where C is independent of n. Thus a function $\hat{G}_y$ which is a solution of

$$\begin{cases} -\Delta \hat{G}_y + \hat{G}_y = \delta_y & \text{in} \quad \Omega_n \cap B_{2R}, \\ \frac{\partial \hat{G}_y}{\partial \nu} = 0 & \text{on} \quad \partial\Omega_n \cap B_{2R}, \\ \hat{G}_y = 0 & \text{on} \quad \Omega_n \cap \partial B_{2R} \end{cases} \tag{21}$$

satisfies

$$\hat{G}_y(x) \le C \log \left(\frac{C}{|x-y|} \right) \tag{22}$$

for any n, $y \in \Omega_n \cap B_R$ and $x \in \Omega_n \cap B_{2R}$.

Using (22), we follow [1]. First, we assume that $|\hat{x}_n - y_n| = O(\hat{r}_n)$. Recalling that

$$\hat{r}_n^{-2} = \frac{L_n}{\lambda_n} c_n^2 e^{2\pi c_n^2},$$

we have

$$\log \frac{C}{|\hat{x}_n - y_n|} \geq \frac{1}{2} \log \left(\frac{L_n}{\lambda_n} c_n^2 e^{2\pi c_n^2} \right) = \frac{1}{2} \left(\log \frac{L_n}{\lambda_n} c_n^2 + 2\pi c_n^2 \right). \tag{23}$$

On the other hand, for $\tilde{\alpha} > 0$, we see that

$$1 = \frac{L_n}{\lambda_n} \int_{\Omega_n} \hat{u}_n^2 e^{2\pi \hat{u}_n^2} dx \leq \frac{L_n}{\lambda_n} c_n^2 e^{\tilde{\alpha} c_n^2} \int_{\Omega_n} e^{(2\pi - \tilde{\alpha}) \hat{u}_n^2} dx.$$

If $\tilde{\alpha}$ is close to 2π, we have

$$0 \leq \log \frac{L_n}{\lambda_n} c_n^2 + \tilde{\alpha} c_n^2 + C$$

for some constant C. Thus, combining (23) and the inequality, we have

$$\log \frac{C}{|\hat{x}_n - y_n|} \geq \frac{1}{2} \left[(2\pi - \tilde{\alpha}) c_n^2 - C \right].$$

Since

$$c_n \hat{u}_n(y_n) \leq c_n^2,$$

it follows that

$$c_n \hat{u}_n(y_n) \leq C \log \left(\frac{C}{|\hat{x}_n - y_n|} \right)$$

for y_n with $|\hat{x}_n - y_n| = O(\hat{r}_n)$.

Next, we assume that $|\hat{x}_n - y_n|/\hat{r}_n \to \infty$. Since $\hat{G}_{y_n}$ is the solution of (21), we have

$$c_n \hat{u}_n(y_n) = \frac{L_n}{\lambda_n} \int_{\Omega_n \cap B_{2R}} \hat{G}_{y_n} c_n \hat{u}_n e^{2\pi \hat{u}_n^2} dx + \int_{\Omega_n \cap \partial B_{2R}} \frac{\partial \hat{G}_{y_n}}{\partial \nu} c_n \hat{u}_n d\sigma. \tag{24}$$

Since $y_n \in \Omega_n \cap B_R$, using Lemma 2.8, we have

$$\left| \int_{\Omega_n \cap \partial B_{2R}} \frac{\partial \hat{G}_{y_n}}{\partial \nu} c_n \hat{u}_n d\sigma \right| \leq C \left| \partial B_{2R} \cap \mathbb{R}_+^2 \right|. \tag{25}$$

Let us set

$$\begin{aligned} \Omega_{1,n} &= (\Omega_n \cap B_{2R}) \setminus \Omega_{n,A}, \\ \Omega_{2,n} &= \Omega_{n,A} \cap B_{|\hat{x}_n - y_n|/2}(y_n), \\ \Omega_{3,n} &= (\Omega_n \cap B_{2R}) \setminus (\Omega_{1,n} \cup \Omega_{2,n}), \end{aligned}$$

where

$$\Omega_{n,A} = \left\{ x \in \Omega_n \cap B_{2R} \, \middle| \hat{u}_n \geq \frac{c_n}{A} \right\}.$$

Applying the techniques of Step 3 in the section 3 in [1], we have

$$\sup_{x \in \Omega_n} |\hat{x}_n - x|^2 \frac{L_n}{\lambda_n} \hat{u}_n^2 e^{2\pi \hat{u}_n^2} \leq C, \tag{26}$$

where C is independent of n.

By Lemma 2.4 and (22), we first compute that

$$\begin{aligned} I_1 &= \frac{L_n}{\lambda_n} \int_{\Omega_{1,n}} \hat{G}_{y_n} c_n \hat{u}_n e^{2\pi \hat{u}_n^2} dx \\ &\leq \frac{L_n}{\lambda_n} \left(\int_{\Omega_{1,n}} \hat{G}_{y_n}^4 dx \right)^{\frac{1}{4}} \left(\int_{\Omega_{1,n}} (c_n \hat{u}_n)^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega_{1,n}} e^{8\pi \hat{u}_n^2} dx \right)^{\frac{1}{4}} \\ &\leq O(c_n^{-1}) \end{aligned} \tag{27}$$

for sufficiently large A.

Next, we deduce by (22) and (26) that

$$\begin{aligned} I_2 &= \frac{L_n}{\lambda_n} \int_{\Omega_{2,n}} \hat{G}_{y_n} c_n \hat{u}_n e^{2\pi \hat{u}_n^2} dx \\ &\leq A \int_{\Omega_{2,n}} \frac{1}{\pi} \log \left(\frac{C}{|x - y_n|} \right) \frac{C}{|\hat{x}_n - x|^2} dx \\ &\leq \frac{AC}{\pi} \frac{2}{|\hat{x}_n - y_n|^2} \int_{B_{|\hat{x}_n - y_n|/2}(y_n)} \log \left(\frac{C}{|x - y_n|} \right) dx \\ &= \frac{AC}{\pi} \omega_{N-1} \int_0^1 \log \left(\frac{2C}{|\hat{x}_n - y_n| r} \right) r dr \\ &\leq C \log \left(\frac{C}{|\hat{x}_n - y_n|} \right) \end{aligned} \tag{28}$$

with some positive constant C.

Finally, we derive by (22) that

$$\begin{aligned} I_3 &= \frac{L_n}{\lambda_n}\int_{\Omega_{3,n}} \hat{G}_{y_n} c_n \hat{u}_n e^{2\pi\hat{u}_n^2}dx \\ &\le C\log\left(\frac{2C}{|\hat{x}_n - y_n|}\right)\frac{L_n}{\lambda_n}\int_{\Omega_{3,n}} c_n\hat{u}_n e^{2\pi\hat{u}_n^2}dx \\ &\le AC\log\left(\frac{2C}{|\hat{x}_n - y_n|}\right)\frac{L_n}{\lambda_n}\int_{\Omega_{3,n}} \hat{u}_n^2 e^{2\pi\hat{u}_n^2}dx \\ &\le AC\log\left(\frac{2C}{|\hat{x}_n - y_n|}\right). \end{aligned} \tag{29}$$

Hence, by (24), (25) and (27)-(29), we have

$$c_n\hat{u}_n(y_n) \le C\log\left(\frac{C}{|\hat{x}_n - y_n|}\right).$$

for y_n with $|\hat{x}_n - y_n|/\hat{r}_n \to \infty$. Consequently, we conclude Lemma 2.10. □

Lemma 2.11. *We have*

$$c_n\hat{u}_n(\Phi(x - \Psi(\hat{x}_n))) \to G_0 \quad \text{in} \quad C^2_{loc}(\overline{\mathbb{R}^2_+}\setminus\{0\}),$$

where $G_0 \in C^2_{loc}(\mathbb{R}^2\setminus\{0\})$ *is the solution of*

$$-\Delta G_0 + G_0 = \delta_0.$$

Proof. By Lemma 2.10 and the regularity theory, we derive Lemma 2.11. □

Lemma 2.12. *It holds that*

$$\limsup_{n\to\infty}\lambda_n E_{2\pi}(u_n) \le \limsup_{n\to\infty}\frac{\lambda_n}{c_n^2 L_n}.$$

Proof. Going back to the computation (14), we have

$$\begin{aligned} \lambda_n E_{2\pi}(u_n) &\le A^2\frac{\lambda_n}{c_n^2 L_n} + \lambda_n\int_\Omega\left(e^{2\pi|u_n^A|^2} - 1\right)dx \\ &= A^2\frac{\lambda_n}{c_n^2 L_n} + \int_{\Omega_n}\left(e^{2\pi|\hat{u}_n^A|^2} - 1\right)dx \end{aligned}$$

for any $A > 1$. We estimate $\int_{\Omega_n} \left(e^{2\pi|\hat{u}_n^A|^2} - 1\right) dx$. We recall that $u_n \in \Sigma_{\lambda_n}$ which implies $\int_{\Omega_n} (|\nabla \hat{u}_n|^2 + \hat{u}_n^2)\, dx = 1$. Then, it follows from Lemma 2.11 that $\hat{u}_n(\Phi(x - \Psi(\hat{x}_n))) \rightharpoonup 0$ weakly in $H^1(B_R \cap \mathbb{R}^2_+)$ for each $R > 0$. The fact and Lemma 2.7 yield that

$$\lim_{n\to\infty} \int_{\Omega_n} \left(e^{2\pi|\hat{u}_n^A|^2} - 1\right) dx = 0$$

for any $A > 1$. Consequently, letting $A \to 1$, we derive that

$$\limsup_{n\to\infty} \lambda_n E_{2\pi}(u_n) \leq \limsup_{n\to\infty} \frac{\lambda_n}{c_n^2 L_n}.$$

□

Lemma 2.13. *It holds that*

$$\limsup_{n\to\infty} \frac{\lambda_n}{c_n^2 L_n} \leq \frac{\pi}{2} e^{4\pi K + 1},$$

where

$$K = \lim_{|x|\to 0} \left(G_0(x) - \frac{1}{2\pi} \log \frac{1}{|x|}\right)$$

and G_0 is a function as in Lemma 2.11.

Proof. We follow [25] (see also Section 4 in [15]). Fix ε small. We consider a function $\tilde{G}$ solution of

$$\begin{cases} -\Delta \tilde{G}_{n,0} = \delta_0 & \text{in} \quad \Omega_n \cap B_\varepsilon, \\ \frac{\partial \tilde{G}_{n,0}}{\partial \nu} = 0 & \text{on} \quad \partial\Omega_n \cap B_\varepsilon, \\ \tilde{G}_{n,0} = \frac{1}{2\pi} \log \frac{1}{\varepsilon} & \text{on} \quad \Omega_n \cap \partial B_\varepsilon. \end{cases}$$

Using a reflection argument, one can obtain the existence of $\tilde{G}_{n,0}$, which can be represented by

$$\tilde{G}_{n,0}(x) = \frac{1}{2\pi} \log\left(\frac{1}{|x|}\right) + w_n(x), \tag{30}$$

where $w_n = O(\varepsilon)$ uniformly with respect to n.

For $c_1 \le c_2$ we define a space of functions by

$$\begin{aligned}
&\Lambda_n(c_1, c_2, a, b)\\
:= &\left\{u \in H^1\left(\left[c_1 \le \tilde{G}_{n,0} \le c_2\right]\right) \;\middle|\; \right.\\
&\left. u = a \text{ on } \left[\tilde{G}_{n,0} = c_1\right],\ u = b \text{ on } \left[\tilde{G}_{n,0} = c_2\right],\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_n \cap B_\varepsilon\right\}.
\end{aligned}$$

It can be seen that $\inf_{u\in\Lambda_n} \int_{\left[c_1 \le \tilde{G}_{n,0} \le c_2\right]} |\nabla u|^2 dx$ is attained by a function $\mathcal{B}$ having the form

$$\mathcal{B} = \frac{b(\tilde{G}_{n,0} - c_1) - a(\tilde{G}_{n,0} - c_2)}{c_2 - c_1}$$

and satisfying

$$\int_{\left[c_1 \le \tilde{G}_{n,0} \le c_2\right]} |\nabla \mathcal{B}|^2 dx = \frac{|b-a|^2}{c_2 - c_1}. \tag{31}$$

Choose $y_n \in \Omega \cap B_\varepsilon$ such that $|y_n| = R_1 \hat{r}_n$ for some large constant R_1. Set

$$\mathcal{S}_n = \left\{x \in \Omega_n \cap B_\varepsilon \;\middle|\; \tilde{G}_{n,0}(x) = \tilde{G}_{n,0}(y_n)\right\}.$$

If $x \in \mathcal{S}_n$, then by (30), we see that

$$|x| = |y_n| e^{2\pi(v_n(x) - v_n(y_n))},$$

which implies the existence of a constant $c > 0$ independent of n such that

$$e^{-c\varepsilon} R_1 \hat{r}_n \le |x| \le e^{c\varepsilon} R_1 \hat{r}_n.$$

Consequently, we get

$$\mathcal{S}_n \subset \Omega_n \cap \left(B_{e^{c\varepsilon} R_1 \hat{r}_n} \setminus B_{e^{-c\varepsilon} R_1 \hat{r}_n}\right).$$

We recall that

$$c_n \left(\hat{u}_n(\Phi(\hat{r}_n z - \Psi(\hat{x}_n))) - c_n\right) \to \eta_0 = -\frac{1}{2\pi} \log\left(1 + \frac{\pi}{2}|z|^2\right) \quad \text{in} \quad C^2_{loc}(\overline{\mathbb{R}^2_+}).$$

By the fact and Lemma 2.11, we have

$$\inf_{x \in \mathcal{S}_n} \hat{u}_n(x) \ge b_n := c_n + \frac{\eta_0(e^{c\varepsilon} R_1) + o_n(R_1)}{c_n} \tag{32}$$

and

$$\sup_{x\in\Omega\cap\partial B_\varepsilon} \hat{u}_n(x) \le a_n := \frac{\sup_{x\in\Omega\cap\partial B_\varepsilon} G_0(x) + o_n(\varepsilon)}{c_n}, \tag{33}$$

where $o_n(R_1) \to 0$, $o_n(\varepsilon) \to 0$ as $n \to \infty$ for fixed R_1, ε, and G_0 is the function as in Lemma 2.11. If n is large, we have $a_n < b_n$. Put $\mathcal{G}_n = \left\{x \in \Omega_n \cap B_\varepsilon \;\middle|\; \tilde{G}_{n,0}(x) > \tilde{G}_{n,0}(y_n)\right\}$, and set $\hat{U}_n = \min\{\max\{\hat{u}_n, a_n\}, b_n\}$. From (32) and (33), we get $\hat{U}_n \in \Lambda_n\left(-(2\pi)^{-1}\log\varepsilon, \tilde{G}_{n,0}(y_n), a_n, b_n\right)$. By (31), we obtain

$$\int_{\mathcal{G}_n} |\nabla \hat{U}_n|^2 dx \ge \frac{b_n - a_n}{\tilde{G}_{n,0}(y_n) + \frac{1}{2\pi}\log\varepsilon}. \tag{34}$$

Notice that

$$B_{e^{-c\varepsilon}R_1\hat{r}_n} \cap \Omega_n \subset \left[\tilde{G}_{n,0} > \tilde{G}_{n,0}(y_n)\right].$$

Taking R_2 large, we get

$$\begin{aligned}
&\int_{\mathcal{G}_n} |\nabla \hat{U}_n|^2 dx \\
\le\;& \int_{\mathcal{G}_n} |\nabla \hat{u}_n|^2 dx \\
\le\;& \int_{\Omega_n\cap B_\varepsilon} |\nabla \hat{u}_n|^2 dx - \int_{B_{e^{-c\varepsilon}R_1\hat{r}_n}} |\nabla \hat{u}_n|^2 dx \\
\le\;& 1 - \int_{\Omega_n\setminus B_\varepsilon} \left(|\nabla \hat{u}_n|^2 + \hat{u}_n^2\right) dx - \int_{B_{e^{-c\varepsilon}R_1\hat{r}_n}} |\nabla \hat{u}_n|^2 dx \\
\le\;& 1 - \int_{(\Omega_n\setminus B_\varepsilon)\cap B_{R_2}} \left(|\nabla \hat{u}_n|^2 + \hat{u}_n^2\right) dx - \int_{B_{e^{-c\varepsilon}R_1\hat{r}_n}} |\nabla \hat{u}_n|^2 dx.
\end{aligned}$$

Following the computations in Section 4 in [15], we derive that

$$\begin{aligned}
\int_{\mathcal{G}_n} |\nabla \hat{U}_n|^2 dx \;\le\;& 1 + \frac{1}{c_n^2}\bigg(\frac{1}{2\pi}\log\frac{\varepsilon}{R_1} - K - \frac{1}{4\pi}\log\frac{\pi}{2} + \frac{1}{4\pi} \\
&+ o_n(1) + o_\varepsilon(1) + o_{R_1}(1) + o_{R_2}(1)\bigg).
\end{aligned}$$

Using (32)-(34) and estimating $\int_{\mathcal{G}_n} |\nabla \hat{U}_n|^2 dx$ from below, we have

$$\frac{1}{4\pi}\frac{\lambda_n}{c_n^2 L_n} \le \frac{1}{4\pi}\log\frac{\pi}{2} + K + \frac{1}{4\pi} + o_n(1) + o_\varepsilon(1) + o_{R_1}(1) + o_{R_2}(1).$$

Letting $n \to \infty$, and then $\varepsilon \to 0$, $R_1 \to \infty$ and $R_2 \to \infty$, we obtain

$$\limsup_{n\to\infty} \frac{\lambda_n}{c_n^2 L_n} \leq \frac{\pi}{2} e^{4\pi K+1}.$$

□

Now we are in a position to prove the boundedness of c_n. By Proposition 2.1 and Lemmas 2.12, 2.13, we derive that

$$I_{2\pi} \leq \liminf_{n\to\infty} \lambda_n E_{2\pi}(u_n) \leq \limsup_{n\to\infty} \lambda_n E_{2\pi}(u_n) \leq \frac{\pi}{2} e^{4\pi K+1}.$$

It is known that $\pi e^{4\pi K+1} < d_{4\pi}$ and $d_{4\pi} = 2I_{2\pi}$. Thus, we derive that

$$\frac{\pi}{2} e^{4\pi K+1} < \liminf_{n\to\infty} \lambda_n E_{2\pi}(u_n) \leq \limsup_{n\to\infty} \lambda_n E_{2\pi}(u_n) \leq \frac{\pi}{2} e^{4\pi K+1},$$

which is a contradiction. Therefore, it holds that $c_n \leq M_2$ for some constant M_2 which is independent of n.

Applying the techniques of [8], we have $c_n \geq M_1$ with a positive constant M_1. Consequently, we complete the proof of Theorem 1.1.

2.2. Proof of Theorem 1.2

Fix $\alpha \in (\alpha_*, 2\pi]$. We assume that $\lambda_n \to \infty$ as $n \to \infty$ and $u_n = u_{\lambda_n}$ is a maximizer of $\sup_{u\in\Sigma_{\lambda_n}} E_\alpha(u)$ for large n. In order to summarize properties of u_n, we set $\hat{u}_n(x) = u_n(x/\sqrt{\lambda_n}+x_n)$ and $\Omega_n := \left\{\sqrt{\lambda_n}(x - x_n) \mid x \in \Omega\right\}$, where x_n is a maximum point of u_n. In Proposition 2.14 below, the uniqueness of maximum point of u_n will be obtained. Then, we write $V_n^* := \Omega_n \cap B_{2\kappa\sqrt{\lambda_n}}$. Under the setting, we have the next proposition.

Proposition 2.14. *We have the following results.*

(I) It holds that

$$M_1 \leq \sup_{x\in\Omega} u_n(x) \leq M_2,$$

where M_1 and M_2 are positive constants independent of n.

(II) For n sufficiently large, u_n has a unique maximum and the maximum point lies on the boundary of Ω.

(III) For any $\varepsilon > 0$, there exist positive constants R and N such that for any $n \geq N$ we have

$$u_n(x) \leq M_3 \varepsilon e^{-\mu_1 \delta(x) \sqrt{\lambda}} \quad \text{for} \quad x \in \overline{\Omega} \setminus B_{R/\sqrt{\lambda_n}}(x_\lambda),$$

where $x_n \in \partial\Omega$ is the unique maximum point of u_n, $\delta(x) = \min\left\{\operatorname{dist}\left(x, \partial B_{R/\sqrt{\lambda_n}}(x_\lambda)\right), \mu_2\right\}$ and M_3, μ_1, μ_2 are positive constants depending only on Ω.

(IV) There exists u_0 which is a maximizer of I_α such that

$$\lim_{n\to\infty} \int_{V_n^*} \left(|\nabla (\hat{u}_n - u_0)|^2 + |\hat{u}_n - u_0|^2\right) dx = 0.$$

(V) There exists a positive constant C such that

$$\hat{u}_n(x) \leq Ce^{-C|x|} \quad \text{for} \quad x \in V_n^*.$$

Proof. If $\alpha \in (\alpha_*, 2\pi)$, (I)-(III) are obtained by [8]. If $\alpha = 2\pi$, by Theorem 1.1 and the techniques in [8], we obtained (I)-(III).

For the proof of (IV), we recall the following convergence in the subsection 2.2 of [8].

$$u_n\left(\Phi_n\left(\frac{z}{\sqrt{\lambda_n}} + x_n\right)\right) \to u_0 \quad \text{in} \quad C^2_{loc}(\overline{\mathbb{R}^2_+}). \tag{35}$$

By (35), we have

$$\begin{aligned}\lim_{n\to\infty} \int_{V_n^*} \left(|\nabla \hat{u}_n|^2 + |\hat{u}_n|^2\right) dx &\geq \lim_{n\to\infty} \int_{\Omega_n \cap B_{2R}} \left(|\nabla \hat{u}_n|^2 + |\hat{u}_n|^2\right) dx \\ &= \int_{B_R \cap \mathbb{R}^2_+} \left(|\nabla u_0|^2 + |u_0|^2\right) dx\end{aligned}$$

for any $R > 0$. Letting $R \to \infty$, we derive that

$$\lim_{n\to\infty} \int_{V_n^*} \left(|\nabla \hat{u}_n|^2 + |\hat{u}_n|^2\right) dx \geq 1.$$

Moreover, since $u_n \in \Sigma_{\lambda_n}$, we have

$$\int_{V_n^*} \left(|\nabla \hat{u}_n|^2 + |\hat{u}_n|^2\right) dx \leq \int_{\Omega_n} \left(|\nabla \hat{u}_n|^2 + |\hat{u}_n|^2\right) dx = 1$$

for any n. Thus, it follows that

$$\lim_{n\to\infty}\int_{V_n^*}\left(|\nabla\hat{u}_n|^2+|\hat{u}_n|^2\right)dx=1. \tag{36}$$

Using (35) again, we observe that

$$\lim_{n\to\infty}\int_{V_n^*}\left(\nabla\hat{u}_n\nabla u_0+\hat{u}_n u_0\right)dx=\int_{\mathbb{R}^2_+}\left(|\nabla u_0|^2+|u_0|^2\right)dx=1. \tag{37}$$

Combining (36) and (37), we have

$$\lim_{n\to\infty}\int_{V_n^*}\left(|\nabla\left(\hat{u}_n-u_0\right)|^2+|\hat{u}_n-u_0|^2\right)dx=0.$$

Hence, (IV) holds.

Finally, we prove (V). Applying the proof of (4.30) in [16], we derive that for any $R>0$ there exists N such that for $n\geq N$ it holds that

$$\sup_{x\in V_n^*\setminus B_R}\hat{u}_n(x)\leq\sup_{x\in V_n^*\cap\partial B_R}\hat{u}_n(x).$$

Hence, by (III), we obtain (V) in the same way as the proof of (3.5) in [16]. Consequently, we conclude (I)-(V). □

We assume that $x_n\to x_0\in\partial\Omega$ after passing to a subsequence. Moreover, after a rotation and a translation n-dependent we may assume that $x_n=0$. Then, Ω can be described in a small ball $B_{2\kappa}(x_n)$ as the set $\{x=(x_1,x_2)\mid x_2>\psi_n(x_1)\}$, where ψ_n is represented by

$$\psi_n(x_2)=\frac{1}{2}H(x_n)x_1^2+o(x_1^2).$$

The set $\Omega\cap B_{2\kappa}(x_n)$ is denoted by V_n. Further, we may also assume that ψ_n converges locally in a C^2-sense to ψ_0, a corresponding parametrization at x_0.

First, we obtain the upper bound of $\lambda_n E_\alpha(u_n)$. We write again $\hat{u}_n(x)=u_n(x/\sqrt{\lambda_n}+x_n)$, $\Omega_n:=\left\{\sqrt{\lambda}_n(x-x_n)\mid x\in\Omega\right\}$ and $V_n^*:=\Omega_n\cap B_{2\kappa\sqrt{\lambda_n}}$. For an open set X and $v\in H^1(X)$, put

$$J_X^1(v):=\int_X\left(|\nabla v|^2+v^2\right)dx$$

and

$$J^2_X(v) := \int_X \left(e^{2\pi v^2} - 1\right) dx.$$

We note that for any function v defined in $V_n^* \cup \left(B_{2\kappa\sqrt{\lambda_n}} \cap \mathbb{R}^2_+\right)$ it holds that

$$J^1_{V_n^*}(v) = J^1_{B_{2\kappa\sqrt{\lambda_n}} \cap \mathbb{R}^2_+}(v) + J^1_{V_n^* \setminus \mathbb{R}^2_+}(v) - J^1_{\left(B_{2\kappa\sqrt{\lambda_n}} \cap \mathbb{R}^2_+\right) \setminus V_n^*}(v) \tag{38}$$

and that

$$J^2_{V_n^*}(v) = J^2_{B_{2\kappa\sqrt{\lambda_n}} \cap \mathbb{R}^2_+}(v) + J^2_{V_n^* \setminus \mathbb{R}^2_+}(v) - J^2_{\left(B_{2\kappa\sqrt{\lambda_n}} \cap \mathbb{R}^2_+\right) \setminus V_n^*}(v). \tag{39}$$

We define a function u_n^* on $V_n^* \cup \left(B_{2\kappa\sqrt{\lambda_n}} \cap \mathbb{R}^2_+\right)$ by

$$u_n^*(x) = \begin{cases} \hat{u}_n(x) & (x \in V_n^*), \\ \hat{u}_n(x_1, \psi_n(x_1/\sqrt{\lambda_n})) & (x \notin V_n^*), \end{cases}$$

and take a function $\tau_n^* \in C^\infty(\mathbb{R}^2)$ such that

$$\tau_n^*(x) = \begin{cases} 1 & \text{if} \quad x \in B_{\kappa\sqrt{\lambda_n}}, \\ 0 & \text{if} \quad x \in \mathbb{R}^2 \setminus B_{2\kappa\sqrt{\lambda_n}}, \end{cases} \qquad |\nabla\tau(x)| \leq \frac{2}{\kappa\sqrt{\lambda_n}}.$$

By (38) and Proposition 2.14 (III), we derive that

$$\begin{aligned} & J^1_{V_n^*}(\hat{u}_n) \\ = \ & J^1_{B_{2\kappa\sqrt{\lambda_n}} \cap \mathbb{R}^2_+}(\tau_n^* u_n^*) + J^1_{V_n^* \setminus \mathbb{R}^2_+}(\tau_n^* u_n^*) - J^1_{\left(B_{2\kappa\sqrt{\lambda_n}} \cap \mathbb{R}^2_+\right) \setminus V_n^*}(\tau_n^* u_n^*) \\ & + O(e^{-c\sqrt{\lambda_n}}) \end{aligned} \tag{40}$$

with some positive constant c. Then, we have the following results:

$$\begin{aligned} \mathcal{J}_1 \ := \ & J^1_{V_n^* \setminus \mathbb{R}^2_+}(\tau_n^* u_n^*) \\ = \ & \int_{-2\kappa\sqrt{\lambda_n}}^{2\kappa\sqrt{\lambda_n}} \left[\int_{\sqrt{\lambda_n}\psi_n^-(x_1/\sqrt{\lambda_n})}^{0} \left(|\nabla(\tau_n^* u_n^*)|^2 + |\tau_n^* u_n^*|^2\right) dx_2\right] dx_1 \\ & + O(e^{-c\sqrt{\lambda_n}}). \end{aligned} \tag{41}$$

$$\begin{aligned} \mathcal{J}_2 \ := \ & J^1_{\left(B_{2\kappa\sqrt{\lambda_n}} \cap \mathbb{R}^2_+\right) \setminus V_n^*}(\tau_n^* u_n^*) \\ = \ & \int_{-2\kappa\sqrt{\lambda_n}}^{2\kappa\sqrt{\lambda_n}} \left[\int_0^{\sqrt{\lambda_n}\psi_n^+(x_1/\sqrt{\lambda_n})} \left(|\nabla(\tau_n^* u_n^*)|^2 + |\tau_n^* u_n^*|^2\right) dx_2\right] dx_1 \\ & + O(e^{-c\sqrt{\lambda_n}}). \end{aligned} \tag{42}$$

By Proposition 2.14 (IV), (V), (41) and (42), applying the dominated convergence theorem, we have

$$\begin{aligned}&\lim_{n\to\infty}\sqrt{\lambda_n}\,(\mathcal{J}_1-\mathcal{J}_2)\\ =\;&-\frac{1}{2}H(x_0)\int_{-\infty}^{+\infty}\left(|\nabla u_0(x_1,0)|^2+|u_0(x_1,0)|^2\right)x_1^2dx_1. \qquad (43)\end{aligned}$$

For simplicity, we write

$$T_1=\int_{-\infty}^{+\infty}\left(|\nabla u_0(x_1,0)|^2+|u_0(x_1,0)|^2\right)x_1^2dx_1.$$

Since $J^1_{V_n^*}(u_n^*)\le 1$, combining (40)-(43), we obtain

$$1\ge J^1_{B_{2\kappa\sqrt{\lambda_n}}\cap\mathbb{R}^2_+}(\tau_n^*u_n^*)-\frac{T_1}{2}H(x_0)\frac{1}{\sqrt{\lambda_n}}+o(\sqrt{\lambda_n}^{-1})$$

or

$$J^1_{B_{2\kappa\sqrt{\lambda_n}}\cap\mathbb{R}^2_+}(\tau_n^*u_n^*)\le 1+\frac{T_1}{2}H(x_0)\frac{1}{\sqrt{\lambda_n}}+o(\sqrt{\lambda_n}^{-1}). \qquad (44)$$

Using (44), we estimate $\lambda_nJ^2_\Omega(u_n)$. By Proposition 2.14 (III) and (39), we see that

$$\begin{aligned}&\lambda_nJ^2_\Omega(u_n)\\ =\;&J^2_{V_n^*}(\hat{u}_n)+O(e^{-c\sqrt{\lambda_n}})\\ =\;&J^2_{B_{2\kappa\sqrt{\lambda_n}}\cap\mathbb{R}^2_+}(\tau_n^*u_n^*)+J^2_{V_n^*\setminus\mathbb{R}^2_+}(\tau_n^*u_n^*)-J^2_{\left(B_{2\kappa\sqrt{\lambda_n}}\cap\mathbb{R}^2_+\right)\setminus V_n^*}(\tau_n^*u_n^*)\\ &+O(e^{-c\sqrt{\lambda_n}}). \qquad (45)\end{aligned}$$

with some positive constant c. Computing in the same way as (41)-(43), we derive that

$$\begin{aligned}&\lim_{n\to\infty}\sqrt{\lambda_n}\left[J^2_{V_n^*\setminus\mathbb{R}^2_+}(\tau_n^*u_n^*)-J^2_{\left(B_{2\kappa\sqrt{\lambda_n}}\cap\mathbb{R}^2_+\right)\setminus V_n^*}(\tau_n^*u_n^*)\right]\\ =\;&-\frac{1}{2}H(x_0)\int_{-\infty}^{+\infty}\left(e^{2\pi|u_0(x_1,0)|^2}-1\right)x_1^2dx. \qquad (46)\end{aligned}$$

Moreover, we have

$$
\begin{aligned}
& J^2_{B_{2\kappa\sqrt{\lambda_n}}\cap\mathbb{R}^2_+}(\tau_n^* u_n^*) \\
\leq & \int_{\mathbb{R}^2_+} \left\{ \exp\left[\left(1 + \frac{T_1}{2} H(x_0) \frac{1}{\sqrt{\lambda_n}} + o(\sqrt{\lambda_n}^{-1}) \right) \frac{|\tau_n^* u_n^*|^2}{J^1_{B_{2\kappa\sqrt{\lambda_n}}\cap\mathbb{R}^2_+}(\tau_n^* u_n^*)} \right] - 1 \right\} dx \\
\leq & I_{2\pi} + \pi T_1 H(x_0) \frac{1}{\sqrt{\lambda_n}} \int_{\mathbb{R}^2_+} u_0^2 e^{2\pi u_0^2} dx + o(\sqrt{\lambda_n}^{-1}). \qquad (47)
\end{aligned}
$$

Thus, (45)-(47) yield

$$
\lambda_n J^2_\Omega(u_n) \leq I_{2\pi} + \frac{T^*}{2} H(x_0) \frac{1}{\sqrt{\lambda_n}} + o(\sqrt{\lambda_n}^{-1}),
$$

where

$$
\begin{aligned}
T^* = & \int_{-\infty}^{+\infty} \left[\left(2\pi \int_{\mathbb{R}^2_+} u_0^2 e^{2\pi u_0^2} dx \right) \left(|\nabla u_0(x_1,0)|^2 + |u_0(x_1,0)|^2 \right) \right. \\
& \qquad \left. - \left(e^{2\pi |u_0(x_1,0)|^2} - 1 \right) \right] x_1^2 dx.
\end{aligned}
$$

Hence, we obtain

$$
\lambda_n E_\alpha(u_n) = \lambda_n J^2_\Omega(u_n) \leq I_{2\pi} + \frac{T^*}{2} H(x_0) \frac{1}{\sqrt{\lambda_n}} + o(\sqrt{\lambda_n}^{-1}).
$$

Here, we prove the positivity of T^*. We recall that u_0 is a maximizer of $I_{2\pi}$, and thus it holds that

$$
-\Delta u_0 + u_0 = \frac{u_0 e^{2\pi u_0^2}}{\int_{\mathbb{R}^2_+} u_0^2 e^{2\pi u_0^2} dx} \quad \text{in} \quad \mathbb{R}^2_+, \quad u_0 \in H^1(\mathbb{R}^2_+).
$$

Multiplying both sides by $x_2^2 \partial u_0 / \partial x_2$ and integrating it on $\mathbb{R}^2_+$, we have

$$
\begin{aligned}
& \int_{\mathbb{R}^2_+} x_2^2 \frac{\partial u_0}{\partial \nu} \left(-\Delta u_0 + u_0 \right) dx - \left(\int_{\mathbb{R}^2_+} u_0^2 e^{2\pi u_0^2} dx \right)^{-1} \int_{\mathbb{R}^2_+} x_2^2 \frac{\partial u_0}{\partial \nu} u_0 e^{2\pi u_0^2} dx \\
= & \ 0
\end{aligned}
$$

By a direct computation in the same way as the proof of Lemma 3.3 in [18], we have $T^* > 0$.

Next, we estimate $\lambda_n E_\alpha(u_n)$ below. Computing $J_{\Omega_n}(\tau_n^* u_0)$ directly, we have

$$\lambda_n J_\Omega^2(\tau_n^* u_0) \geq I_{2\pi} + \frac{T^*}{2} H(x_0) \frac{1}{\sqrt{\lambda_n}} + o(\sqrt{\lambda_n}^{-1}).$$

Thus,

$$\lambda_n E_\alpha(u_n) \geq I_{2\pi} + \frac{T^*}{2} H(x_0) \frac{1}{\sqrt{\lambda_n}} + o(\sqrt{\lambda_n}^{-1}).$$

Consequently, we obtain the following energy expansion

$$\lambda_n E_{2\pi}(u_n) = I_{2\pi} + \frac{T^*}{2} H(x_0) \frac{1}{\sqrt{\lambda_n}} + o(\sqrt{\lambda_n}^{-1}).$$

Then we have

$$\lim_{n\to\infty} H(x_n) = \max_{x\in\partial\Omega} H(x),$$

which completes the proof of Theorem 1.2.

3. Maximizer for $\sup_{u\in\Sigma_\lambda^0} E_\alpha(u)$: Proofs of Theorems 1.3 and 1.4

We fix $\alpha \in (0, 4\pi]$ and assume that λ_n is a sequence with $\lambda_n \to \infty$ as $n \to \infty$. Suppose that $v_n := v_{\lambda_n}$ is a maximizer for $\sup_{u\in\Sigma_{\lambda_n}^0} E_\alpha(u)$. Assume that $x_n \in \Omega$ is a maximum point of v_n and set $\hat{v}_n(x) = v_n(x/\sqrt{\lambda_n} + x_n)$.

First, we check that $\lim_{n\to\infty} \lambda_n E_\alpha(v_n) = d_\alpha$, and thus, $\hat{v}_n$ is a maximizing sequence of d_α. We take a positive constant R and $x_0 \in \Omega$ satisfying $B_{2R}(x_0) \subset \Omega$. Then, we define a function $\tau_R \in C^\infty(\mathbb{R}^2)$ by

$$\tau_R(x) = \begin{cases} 1 & \text{if} \quad x \in B_R, \\ 0 & \text{if} \quad x \in \mathbb{R}^2 \setminus B_{2R}, \end{cases} \qquad |\nabla \tau(x)| \leq C,$$

and we set $\tau_{R,n}(x) = \tau_R\left((x - x_0)/\sqrt{\lambda_n}\right)$. For any $\psi \in H^1(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} (|\nabla\psi|^2 + \psi^2)\, dx = 1$, we see that

$$M_n := \int_{\mathbb{R}^2} \left(|\nabla(\tau_{R,n}\psi)|^2 + |\tau_{R,n}\psi|^2\right) dx = 1 + o(1).$$

Thus, for $R^* > 0$ with $R^* \leq 2R\sqrt{\lambda_n}$, we have

$$\begin{aligned} \int_{B_{R^*}(x_0)} \left(e^{\alpha\psi^2} - 1\right) dx &\leq \int_{B_{2R\sqrt{\lambda_n}}(x_0)} \left(e^{\alpha\frac{(\tau_{R,n}\psi)^2}{M_n}} - 1\right) dx + o(1) \\ &\leq \lambda_n E_\alpha(v_n) + o(1). \end{aligned}$$

Letting $n \to \infty$, and then $R^* \to \infty$, we derive that

$$\int_{\mathbb{R}^2} \left(e^{\alpha\psi^2} - 1 \right) dx \le \liminf_{n\to\infty} \lambda_n E_\alpha(v_n).$$

Hence, it holds that $d_\alpha \le \liminf_{n\to\infty} \lambda_n E_\alpha(v_n)$. On the other hand, by extending v_n by 0 outside Ω, it holds that $\lambda_n E_\alpha(v_n) \le d_\alpha$ for any n. Hence, we obtain that $\lim_{n\to\infty} \lambda_n E_\alpha(v_n) = d_\alpha$ and $\hat{v}_n$ is a maximizing sequence of d_α.

3.1. Proof of Theorem 1.3

In the case $\alpha < 4\pi$, we derive (I) and (II) by applying the techniques in [8]. In the case $\alpha = 4\pi$, we first obtain that $\sup_{x\in\Omega} v_n(x) \le M_2$ for some positive constant M_2 independent of n. The proof follows Section 2. Then, we prove (I) in the same way as the proof of Theorem 1.1 (I) in [8]. In this case, the theorem is also obtained by Theorem 1.2 in [10] and the fact that $\hat{v}_n$ is a maximizing sequence of d_α.

3.2. Proof of Theorem 1.4

We fix $\alpha \in (\beta_*, 4\pi]$. In order to prove Theorem 1.4, we summarize the properties of v_n.

Proposition 3.1. *We have the following results.*

(I) It holds that

$$M_1 \le \sup_{x\in\Omega} v_n(x) \le M_2,$$

where M_1 and M_2 are positive constants independent of n.

(II) For n sufficiently large v_n has a unique maximum at $x_n \in \Omega$, and it holds that

$$\lim_{n\to\infty} \sqrt{\lambda_n} \text{dist}(x_n, \partial\Omega) = \infty.$$

(III) There exists v_0 which is a maximizer of d_α such that

$$\hat{v}_n \to v_0 \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2),$$

where $\hat{v}_n(x) = v_n(x/\sqrt{\lambda_n} + x_n)$.

We may assume that, up to a subsequence, $x_n \to x_0 \in \overline{\Omega}$ as $n \to \infty$. Then,

$$d_n := \mathrm{dist}(x_n, \partial\Omega) \to d_0 := \mathrm{dist}(x_0, \partial\Omega).$$

Set $\Omega_n = \left\{ \sqrt{\lambda_n}(x - x_n) \;\middle|\; x \in \Omega \right\}$ and $\hat{v}_n = v_n(x/\sqrt{\lambda_n} + x_n)$. The function $\hat{v}_n$ satisfies

$$\begin{cases} -\Delta \hat{v}_n + \hat{v}_n = L_n \hat{v}_n e^{\alpha \hat{v}_n^2} & \text{in} \quad \Omega_n, \\ \hat{v}_n = 0 & \text{on} \quad \partial\Omega_n, \end{cases}$$

where $L_n = \left(\lambda_n \int_\Omega v_n^2 e^{\alpha v_n^2} dx \right)^{-1}$. We note that $L_n \le 1 + \lambda_1(\Omega)/\lambda_n$ holds for any n, where $\lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ with the zero Dirichlet boundary condition on Ω. The function v_0 in Proposition 3.1 (III) satisfies

$$-\Delta v_0 + v_0 = \frac{v_0 e^{\alpha v_0^2}}{\int_{\mathbb{R}^2} v_0^2 e^{\alpha v_0^2} dx} \quad \text{in} \quad \mathbb{R}^2$$

or

$$-\Delta v_0 + L_\infty v_0 = (1 - L_\infty) v_0 (e^{\alpha v_0^2} - 1) \quad \text{in} \quad \mathbb{R}^2,$$

where $L_\infty = 1 - \left(\int_{\mathbb{R}^2} v_0^2 e^{\alpha v_0^2} dx \right)^{-1}$. By the Pohozaev identity, we have $L_\infty \in (0, 1)$. It follows from the upper bound of L_n and Proposition 3.1 that $1 - L_n \to L_\infty$ as $n \to \infty$. We define a constant as

$$\mathscr{L} := \max \left\{ 1 - \frac{1}{\int_{\mathbb{R}^2} v_0^2 e^{\alpha v_0^2} dx} \;\middle|\; v_0 \text{ is a maximizer of } d_\alpha \right\}.$$

We note that by the precompactness of maximizers of d_α, there exists $v_0^* \in H^1(\mathbb{R}^2)$ such that

$$1 - \frac{1}{\int_{\mathbb{R}^2} |v_0^*|^2 e^{\alpha |v_0^*|^2} dx} = \mathscr{L}. \tag{48}$$

We first prepare the following lemma.

Lemma 3.2. *Let K and c be positive constants and let f be a positive function such that $f(r) \to 0$ as $r \to \infty$. For a positive constant ρ and $R \in (0, \rho - 1)$, assume that w_ρ is a solution of*

$$\begin{cases} -w'' - \frac{1}{r} w' + K w = f w & \text{in} \quad (R, \rho), \\ w(R) = c, \quad w(\rho) = 0 \end{cases}$$

and that w_∞ is a solution of

$$\begin{cases} -w'' - \frac{1}{r}w' + Kw = fw \quad \text{in} \quad (R,\infty), \\ w(R) = c, \quad w(\infty) = 0. \end{cases}$$

Then, for large ρ and R, there exists $\varepsilon > 0$ such that $\varepsilon \to 0$ as $R \to \infty$,

$$e^{-\rho(\sqrt{K}+\varepsilon)} \le w_\rho(\rho - 1) \le e^{-\rho(\sqrt{K}-\varepsilon)} \tag{49}$$

and

$$e^{-\rho(\sqrt{K}+\varepsilon)} \le w_\infty(\rho - 1) \le e^{-\rho(\sqrt{K}-\varepsilon)}. \tag{50}$$

Proof. Since the proof of (50) is same as the proof of (49), we only prove (49). Fix large ρ and R with $\rho - 1 > R$. We take small ε_1 such that $K - f(R) \ge (\sqrt{K} - \varepsilon_1)^2$. Then, we consider the equation

$$\begin{cases} -w'' + \left(\sqrt{K} - \varepsilon_1\right)^2 w = 0 \quad \text{in} \quad (R,\rho), \\ w(R) = c, \quad w(\rho) = 0. \end{cases}$$

The solution of the above equation is a supersolution of w_ρ, and thus

$$w_\rho(\rho - 1) \le e^{-\rho(\sqrt{K}-\varepsilon_1)}.$$

For the lower bound, we choose ε_2 such that $\left(R^{-1} + \sqrt{R^{-2} + 4K}\right)/2 < \sqrt{K} + \varepsilon_2$. We consider the equation

$$\begin{cases} -w'' - \frac{1}{R}w' + Kw = 0 \quad \text{in} \quad (R,\rho), \\ w(R) = c, \quad w(\rho) = 0. \end{cases}$$

Since the solution of the equation is a subsolution of w_ρ, by a direct computation, we have

$$w_\rho(\rho - 1) \ge e^{-\rho(\sqrt{K}+\varepsilon_2)}.$$

Hence, taking $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$, we derive (49). By appropriate choices of ε_1 and ε_2, it holds that $\varepsilon \to 0$ as $R \to \infty$. □

By Proposition 3.1 (III) and Lemma 3.2, we have

$$\hat{v}_n(x) \le e^{-|x|\left(\sqrt{L_\infty}+o(1)\right)} \tag{51}$$

for $|x| \geq \rho$ with large ρ and in particular, we have

$$\hat{v}_n(x) = e^{-|x|(\sqrt{L_\infty}+o(1))} \tag{52}$$

for $\rho \leq |x| \leq d_n\sqrt{\lambda_n} - 1$. Moreover, it holds that

$$\frac{\partial \hat{v}_n}{\partial \nu}(x_n^*) \leq -e^{-d_0\sqrt{\lambda_n}(\sqrt{L_\infty}+o(1))},$$

where $x_n^* \in \partial\Omega_n$ satisfies $|x_n^*| = d_n\sqrt{\lambda_n}$. Thus, using Proposition 3.1 (III) and considering suitable ordinary differential equation, we have

$$\frac{\partial \hat{v}_n}{\partial \nu}(x) \leq -e^{-d_0\sqrt{\lambda_n}(\sqrt{L_\infty}+o(1))} \quad \text{for} \quad x \in \partial\Omega_n \cap B_\kappa(x_n^*) \tag{53}$$

with $\kappa > 0$ independent of n. By (51) and the regularity theory, we have

$$\int_{\Omega_n \setminus \overline{B_\rho}} \left(|\nabla \hat{v}_n|^2 + \hat{v}_n^2\right) dx \leq e^{-2\rho(\sqrt{L_\infty}+o(1))}. \tag{54}$$

Set

$$d_\infty = \max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega) = \operatorname{dist}(x_\infty, \partial\Omega).$$

We prove the following lower estimate of $\lambda_n E_\alpha(v_n)$.

Proposition 3.3. *It holds that*

$$\lambda_n E_\alpha(v_n) \geq d_\alpha - e^{-2d_\infty\sqrt{\lambda_n}(\sqrt{\mathscr{L}}+o(1))}$$

as $n \to \infty$.

Proof. We first consider a lower estimate of

$$D(\alpha, \rho) = \sup_{\substack{u\in H_0^1(B_\rho) \\ \int_{B_\rho}\left(|\nabla u|^2+u^2\right)dx=1}} \int_{B_\rho} \left(e^{\alpha u^2} - 1\right) dx$$

as $\rho \to \infty$. We take v_0^* a maximizer of d_α satisfying (48). The function v_0^* is a solution of

$$\begin{cases} -w'' - \frac{1}{r}w' + \mathscr{L}w = (1-\mathscr{L})w\left(e^{\alpha w^2} - 1\right) \quad \text{in} \quad (0, \infty), \\ w'(0) = 0, \quad w(\infty) = 0. \end{cases}$$

Thus, by Lemma 3.2, we have

$$v_0^*(\rho-1) = e^{-\rho\left(\sqrt{\mathscr{L}}+o(1)\right)} \tag{55}$$

as $\rho \to \infty$. Let Ψ_ρ be a solution of

$$\begin{cases} -\Delta\psi + \mathscr{L}\psi = 0 & \text{in} \quad B_\rho \setminus \overline{B_{\rho-1}}, \\ \psi = v_0^* & \text{on} \quad \partial B_{\rho-1}, \\ \psi = 0 & \text{on} \quad \partial B_\rho. \end{cases} \tag{56}$$

By (55), (56) and the regularity theory, we observe that

$$\int_{\mathbb{R}^2\setminus\overline{B_{\rho-1}}} \left(|\nabla v_0^*|^2 + |v_0^*|^2\right) dx = e^{-2\rho(\sqrt{\mathscr{L}}+o(1))} \tag{57}$$

and

$$\int_{B_\rho\setminus\overline{B_{\rho-1}}} \left(|\nabla \Psi_\rho|^2 + \Psi_\rho^2\right) dx = e^{-2\rho(\sqrt{\mathscr{L}}+o(1))}. \tag{58}$$

Define a function $\underline{v}_\rho$ by

$$\underline{v}_\rho(x) = \begin{cases} v_0^*(x) & (|x| \le \rho-1), \\ \Psi_\rho(x) & (\rho-1 \le |x| \le \rho). \end{cases}$$

Then, we have

$$\begin{aligned} & \int_{B_\rho} \left(|\nabla \underline{v}_\rho|^2 + |\underline{v}_\rho|^2\right) dx \\ = \ & \int_{\mathbb{R}^2} \left(|\nabla v_0^*|^2 + |v_0^*|^2\right) dx + \int_{B_\rho\setminus\overline{B_{\rho-1}}} \left(|\nabla \Psi_\rho|^2 + \Psi_\rho^2\right) dx \\ & - \int_{\mathbb{R}^2\setminus\overline{B_{\rho-1}}} \left(|\nabla v_0^*|^2 + |v_0^*|^2\right) dx \\ = \ & 1 + \int_{B_\rho\setminus\overline{B_{\rho-1}}} \left(|\nabla \Psi_\rho|^2 + \Psi_\rho^2\right) dx - \int_{\mathbb{R}^2\setminus\overline{B_{\rho-1}}} \left(|\nabla v_0^*|^2 + |v_0^*|^2\right) dx \\ =: \ & 1 + T_1 - T_2. \end{aligned}$$

Using this equality, we have

$$
\begin{aligned}
& D(\alpha,\rho) \\
\geq\ & \int_{B_\rho}\left\{\exp\left[\alpha\frac{\underline{v}_\rho^2}{\int_{B_\rho}\left(|\nabla\underline{v}_\rho|^2+|\underline{v}_\rho|^2\right)dx}\right]-1\right\}dx \\
=\ & \int_{B_\rho}\left[e^{\alpha\left(1+\frac{T_2-T_1}{1+T_1-T_2}\right)\underline{v}_\rho^2}-1\right]dx \\
\geq\ & \int_{B_\rho}\left(e^{\alpha\underline{v}_\rho^2}-1\right)dx+\alpha\frac{T_2-T_1}{1+T_1-T_2}\int_{B_\rho}\underline{v}_\rho^2e^{\alpha\underline{v}_\rho^2}dx \\
=\ & d_\alpha+\int_{B_\rho\setminus\overline{B_{\rho-1}}}\left(e^{\alpha\Psi_\rho^2}-1\right)dx-\int_{\mathbb{R}^2\setminus\overline{B_{\rho-1}}}\left(e^{\alpha|v_0^*|^2}-1\right)dx \\
& +\alpha(T_2-T_1)\int_{B_\rho}\underline{v}_\rho^2e^{\alpha\underline{v}_\rho^2}dx+O\left((T_2-T_1)^2\right). \qquad (59)
\end{aligned}
$$

By (57), we see that

$$
\begin{aligned}
& -\int_{\mathbb{R}^2\setminus\overline{B_{\rho-1}}}\left(e^{\alpha|v_0^*|^2}-1\right)dx+\alpha T_2\int_{B_\rho}\underline{v}_\rho^2e^{\alpha\underline{v}_\rho^2}dx \\
\geq\ & \frac{\alpha}{\mathscr{L}+o_\rho(1)}\left[\int_{\mathbb{R}^2\setminus\overline{B_{\rho-1}}}\left(|\nabla v_0^*|^2+|v_0^*|^2\right)dx-(\mathscr{L}+o_\rho(1))\int_{\mathbb{R}^2\setminus\overline{B_{\rho-1}}}|v_0^*|^2dx\right] \\
& +O\left(\int_{\mathbb{R}^2\setminus B_{\rho-1}}|v_0^*|^4dx\right) \\
\geq\ & O\left(e^{-4\rho(\sqrt{\mathscr{L}}+o(1))}\right). \qquad (60)
\end{aligned}
$$

Moreover, by (58), we have

$$
\begin{aligned}
& \int_{B_\rho\setminus\overline{B_{\rho-1}}}\left(e^{\alpha\Psi_\rho^2}-1\right)dx-\alpha T_1\int_{B_\rho}\underline{v}_\rho^2e^{\alpha\underline{v}_\rho^2}dx \\
\geq\ & -\frac{\alpha}{\mathscr{L}+o_\rho(1)}\int_{B_\rho\setminus\overline{B_{\rho-1}}}\left[|\nabla\Psi_\rho|^2+(\mathscr{L}+o_\rho(1))\,\Psi_\rho^2\right]dx \\
=\ & -\frac{\alpha}{\mathscr{L}+o_\rho(1)}e^{-2\rho(\sqrt{\mathscr{L}}+o(1))} \\
=\ & -e^{-2\rho(\sqrt{\mathscr{L}}+o(1))}. \qquad (61)
\end{aligned}
$$

Hence, (57)-(61) yield

$$D(\alpha,\rho)\geq d_\alpha-e^{-2\rho(\sqrt{\mathscr{L}}+o(1))} \qquad (62)$$

as $\rho \to \infty$.

Using (62), we estimate of $\lambda_n E_\alpha(v_n)$ from below. Since we may assume that $H_0^1(B_{d_\infty}(x_\infty)) \subset H_0^1(\Omega)$, by the scaling, we have

$$\lambda_n E_\alpha(v_n) \geq D(\alpha, d_\infty \sqrt{\lambda_n}).$$

Consequently, the inequality and (62) yield that

$$\lambda_n E_\alpha(v_n) \geq d_\alpha - e^{-2d_\infty \sqrt{\lambda_n}(\sqrt{\mathscr{L}}+o(1))}$$

and complete the proof of Proposition 3.3. □

Next, we prove the following upper estimate of $\lambda_n E_\alpha(v_n)$.

Proposition 3.4. *It follows that*

$$\lambda_n E_\alpha(v_n) \leq d_\alpha - e^{-2d_0 \sqrt{\lambda_n}(\sqrt{L_\infty}+o(1))}$$

as $n \to \infty$.

Proof. Let Φ_n be a solution of

$$\begin{cases} -\Delta\phi + (1 - L_n)\phi = f\phi & \text{in} \quad \mathbb{R}^2 \setminus \overline{B_{d_n\sqrt{\lambda_n}-1}}, \\ \phi = \hat{v}_n & \text{on} \quad \partial B_{d_n\sqrt{\lambda_n}-1}, \\ \phi(x) \to 0 & \text{as} \quad |x| \to \infty. \end{cases}$$

where f is a rapidly decreasing function as $|x| \to \infty$. By (52), applying Lemma 3.2 and the regularity theory, we have

$$\Phi_n(x) = e^{-|x|(\sqrt{L_\infty}+o(1))} \tag{63}$$

for $|x| \geq d_n\sqrt{\lambda_n} - 1$ and

$$\int_{\mathbb{R}^2 \setminus \overline{B_{d_n\sqrt{\lambda_n}-1}}} \left(|\nabla\Phi_n|^2 + \Phi_n^2\right) dx = e^{-2d_n\sqrt{\lambda_n}(\sqrt{L_\infty}+o(1))}. \tag{64}$$

Set

$$\overline{v}_n(x) = \begin{cases} \hat{v}_n(x) & (|x| \leq d_n\sqrt{\lambda_n} - 1), \\ \Phi_n(x) & (|x| \geq d_n\sqrt{\lambda_n} - 1). \end{cases}$$

It follows that

$$\begin{aligned}
1 &= \int_{\Omega_n} \left(|\nabla \hat{v}_n|^2 + \hat{v}_n^2\right) dx \\
&= \int_{\mathbb{R}^2} \left(|\nabla \overline{v}_n|^2 + \overline{v}_n^2\right) dx + \int_{\Omega_n \setminus \overline{B_{d_n\sqrt{\lambda_n}-1}}} \left(|\nabla \hat{v}_n|^2 + \hat{v}_n^2\right) dx \\
&\quad - \int_{\mathbb{R}^2 \setminus \overline{B_{d_n\sqrt{\lambda_n}-1}}} \left(|\nabla \Phi_n|^2 + \Phi_n^2\right) dx \\
&=: \tilde{T}_1 + \tilde{T}_2 - \tilde{T}_3.
\end{aligned}$$

Then, using (54) and (64), we have

$$\begin{aligned}
& \lambda_n E_\alpha(v_n) \\
= & \int_{\Omega_n} \left(e^{\alpha \hat{v}_n^2} - 1\right) dx \\
= & \int_{\Omega_n} \left(e^{\alpha \frac{\hat{v}_n^2}{\tilde{T}_1}} e^{\alpha \frac{\tilde{T}_3 - \tilde{T}_2}{\tilde{T}_1} \hat{v}_n^2} - 1\right) dx \\
= & \int_{\Omega_n} \left(e^{\alpha \frac{\hat{v}_n^2}{\tilde{T}_1}} - 1\right) dx + \alpha \frac{\tilde{T}_3 - \tilde{T}_2}{\tilde{T}_1} \int_{\Omega_n} \hat{v}_n^2 e^{\alpha \frac{\hat{v}_n^2}{\tilde{T}_1}} dx + O\left(\left(\tilde{T}_3 - \tilde{T}_2\right)^2\right) \\
= & \int_{\Omega_n} \left(e^{\alpha \frac{\hat{v}_n^2}{\tilde{T}_1}} - 1\right) dx + \alpha \left(\tilde{T}_3 - \tilde{T}_2\right) \int_{\Omega_n} \hat{v}_n^2 e^{\alpha \hat{v}_n^2} dx + O\left(e^{-4d_0\sqrt{\lambda_n}(\sqrt{L_\infty}+o(1))}\right) \\
= & \int_{\mathbb{R}^2} \left(e^{\alpha \frac{\overline{v}_n^2}{\tilde{T}_1}} - 1\right) dx + \int_{\Omega_n \setminus \overline{B_{d_n\sqrt{\lambda_n}-1}}} \left(e^{\alpha \frac{\hat{v}_n^2}{\tilde{T}_1}} - 1\right) dx - \int_{\mathbb{R}^2 \setminus \overline{B_{d_n\sqrt{\lambda_n}-1}}} \left(e^{\alpha \frac{\Phi_n^2}{\tilde{T}_1}} - 1\right) dx \\
& + \alpha \left(\tilde{T}_3 - \tilde{T}_2\right) \int_{B_\rho} \hat{v}_n^2 e^{\alpha \hat{v}_n^2} dx + O\left(e^{-4d_0\sqrt{\lambda_n}(\sqrt{L_\infty}+o(1))}\right) \\
\leq & \; d_\alpha + \int_{\Omega_n \setminus \overline{B_{d_n\sqrt{\lambda_n}-1}}} \left(e^{\alpha \hat{v}_n^2} - 1\right) dx - \int_{\mathbb{R}^2 \setminus \overline{B_{d_n\sqrt{\lambda_n}-1}}} \left(e^{\alpha \Phi_n^2} - 1\right) dx \\
& + \alpha \left(\tilde{T}_3 - \tilde{T}_2\right) \int_{\Omega_n} \hat{v}_n^2 e^{\alpha \hat{v}_n^2} dx + O\left(e^{-4d_0\sqrt{\lambda_n}(\sqrt{L_\infty}+o(1))}\right). \qquad (65)
\end{aligned}$$

We derive that

$$
\begin{aligned}
& \int_{\Omega_n \setminus \overline{B_{d_n\sqrt{\lambda_n}-1}}} \left(e^{\alpha \hat{v}_n^2} - 1\right) dx - \alpha \tilde{T}_2 \int_{B_\rho} \hat{v}_n^2 e^{\alpha \hat{v}_n^2} dx \\
= \quad & -\frac{\alpha}{1 - L_\infty + o(1)} \int_{\Omega_n \setminus \overline{B_{d_n\sqrt{\lambda_n}-1}}} \left[|\nabla \hat{v}_n|^2 + (L_\infty + o(1)) \hat{v}_n^2 \right] dx \\
& + O\left(\int_{\Omega_n \setminus \overline{B_{d_n\sqrt{\lambda_n}-1}}} \hat{v}_n^4 dx \right) \\
= \quad & -\frac{\alpha}{1 - L_\infty + o(1)} \int_{\partial B_{d_n\sqrt{\lambda_n}-1}} \left(-\frac{\partial \hat{v}_n}{\partial \nu} \right) \hat{v}_n d\sigma \\
& + O\left(e^{-4d_0\sqrt{\lambda_n}(\sqrt{L_\infty}+o(1))} \right)
\end{aligned}
\tag{66}
$$

and that

$$
\begin{aligned}
& -\int_{\mathbb{R}^2 \setminus \overline{B_{d_n\sqrt{\lambda_n}-1}}} \left(e^{\alpha \Psi_n^2} - 1\right) dx + \alpha \tilde{T}_3 \int_{\Omega_n} \hat{v}_n^2 e^{\alpha \hat{v}_n^2} dx \\
= \quad & \frac{\alpha}{1 - L_\infty + o(1)} \int_{\mathbb{R}^2 \setminus \overline{B_{d_n\sqrt{\lambda_n}-1}}} \left[|\nabla \Phi_n|^2 + (L_\infty + o(1)) \Phi_n^2 \right] dx \\
= \quad & \frac{\alpha}{1 - L_\infty + o(1)} \int_{\partial B_{d_n\sqrt{\lambda_n}-1}} \left(-\frac{\partial \Phi_n}{\partial \nu} \right) \Phi_n d\sigma \\
& + O\left(e^{-4d_0\sqrt{\lambda_n}(\sqrt{L_\infty}+o(1))} \right).
\end{aligned}
\tag{67}
$$

Combining (65)-(67), we have

$$
\begin{aligned}
\lambda_n E_\alpha(v_n) \quad \leq \quad & d_\alpha + \frac{\alpha}{1 - L_\infty + o(1)} \int_{\partial B_{d_n\sqrt{\lambda_n}-1}} \left(\frac{\partial \hat{v}_n}{\partial \nu} \hat{v}_n - \frac{\partial \Phi_n}{\partial \nu} \Phi_n \right) d\sigma \\
& + O\left(e^{-4d_0\sqrt{\lambda_n}(\sqrt{L_\infty}+o(1))} \right)
\end{aligned}
$$

Since $\hat{v}_n = \Phi_n$ on $\partial B_{d_n\sqrt{\lambda_n}-1}$, by (53), (63) and the Hopf boundary lemma, we have

$$\begin{aligned}
&\int_{\partial B_{d_n\sqrt{\lambda_n}-1}} \left(\frac{\partial \hat{v}_n}{\partial \nu}\hat{v}_n - \frac{\partial \Phi_n}{\partial \nu}\Phi_n\right) d\sigma \\
= &\int_{\partial B_{d_n\sqrt{\lambda_n}-1}} \left(\frac{\partial \hat{v}_n}{\partial \nu}\Phi_n - \frac{\partial \Phi_n}{\partial \nu}\hat{v}_n\right) d\sigma \\
= &\int_{\partial \Omega_n} \frac{\partial \hat{v}_n}{\partial \nu}\Phi_n d\sigma - \int_{\Omega_n \setminus B_{d_n\sqrt{\lambda_n}-1}} \Delta \hat{v}_n \Phi_n dx + \int_{\mathbb{R}^2 \setminus B_{d_n\sqrt{\lambda_n}-1}} \Delta \Phi_n v_n dx \\
= &\int_{\partial \Omega_n} \frac{\partial \hat{v}_n}{\partial \nu}\Phi_n d\sigma + O\left(e^{-4d_0\sqrt{\lambda_n}(\sqrt{L_\infty}+o(1))}\right) \\
= &-e^{-2d_0\sqrt{\lambda_n}(L_\infty+o(1))} + O\left(e^{-4d_n\sqrt{\lambda_n}(\sqrt{L_\infty}+o(1))}\right).
\end{aligned}$$

Hence, we obtain the upper estimate

$$\lambda_n E_\alpha(v_n) \le d_\alpha - e^{-2d_0\sqrt{\lambda_n}(L_\infty+o(1))}.$$

Consequently, we conclude Proposition 3.4. □

Finally, Proposition 3.3 and 3.4 yield that $L_\infty = \mathscr{L}$ and $d_0 = d_\infty$, which complete the proof of Theorem 1.4.

Acknowledgement

This work was supported by JSPS KAKENHI Grant Number 19K14571. This work was partly supported by Osaka Central Advanced Mathematical Institute: MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849.

References

[1] Adimurthi, O. Druet, Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality. (English summary) Comm. Partial Differential Equations 29 (2004), no. 1-2, 295-322.

[2] L. Carleson, S.-Y. A. Chang, On the existence of an extremal function for an inequality of J. Moser. (French summary) Bull. Sci. Math. (2) 110 (1986), no. 2, 113-127.

[3] W. X. Chen, C. Li, Classification of solutions of some nonlinear elliptic equations. Duke Math. J. 63 (1991), no. 3, 615-622.

[4] M. del Pino, P. L. Felmer, Spike-layered solutions of singularly perturbed elliptic problems in a degenerate setting. (English summary) Indiana Univ. Math. J. 48 (1999), no. 3, 883-898.

[5] P. C. Fife, Semilinear elliptic boundary value problems with small parameters. Arch. Rational Mech. Anal. 52 (1973), 205-232.

[6] M. Flucher, Extremal functions for the Trudinger-Moser inequality in 2 dimensions. Comment. Math. Helv. 67 (1992), no. 3, 471-497.

[7] D. Gilbarg, N. Trudinger, Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001. xiv+517 pp. ISBN: 3-540-41160-7

[8] M. Hashizume, Asymptotic properties of critical points for subcritical Trudinger-Moser functional, OCAMI Preprint Series 2021, 21-5.

[9] M. Ishiwata, Existence and nonexistence of maximizers for variational problems associated with Trudinger-Moser type inequalities in $\mathbb{R}^N$. (English summary) Math. Ann. 351 (2011), no. 4, 781-804.

[10] M. Ishiwata, H. Wadade, Vanishing-concentration-compactness alternative for critical Sobolev embedding with a general integrand in $\mathbb{R}^2$. (English summary) Calc. Var. Partial Differential Equations 60 (2021), no. 6, Paper No. 203, 26 pp.

[11] Y. Li, B. Ruf, A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^n$. (English summary) Indiana Univ. Math. J. 57 (2008), no. 1, 451-480.

[12] C.-S. Lin, W.-M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis system. J. Differential Equations 72 (1988), no. 1, 1-27.

[13] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. I. Rev. Mat. Iberoamericana 1 (1985), no. 1, 145-201.

[14] J. Moser, A sharp form of an inequality by N. Trudinger. Indiana Univ. Math. J. 20 (1970/71), 1077-1092.

[15] V. H. Nguyen, Improved Moser-Trudinger inequality for functions with mean value zero in $\mathbb{R}^n$ and its extremal functions. (English summary) Nonlinear Anal. 163 (2017), 127-145.

[16] W.-M. Ni, I. Takagi, Locating the peaks of least-energy solutions to a semilinear Neumann problem. Duke Math. J. 70 (1993), no. 2, 247-281.

[17] W.-M. Ni, I. Takagi, On the Neumann problem for some semilinear elliptic equations and systems of activator-inhibitor type. Trans. Amer. Math. Soc. 297 (1986), no. 1, 351-368.

[18] W.-M. Ni, I. Takagi, On the shape of least-energy solutions to a semilinear Neumann problem. Comm. Pure Appl. Math. 44 (1991), no. 7, 819-851.

[19] W.-M. Ni, J. Wei, On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems. Comm. Pure Appl. Math. 48 (1995), no. 7, 731-768.

[20] S. I. Pohozaev, The Sobolev Embedding in the Case $pl = n$, Proc. Tech. Sci. Conf. on Adv. Sci. Research 1964-1965, Mathematics Section, Moskov. Ènerget. Inst. Moscow, 1965, 158-170.

[21] B. Ruf, A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$. (English summary) J. Funct. Anal. 219 (2005), no. 2, 340-367.

[22] B. Ruf, F. Sani, Ground states for elliptic equations in $\mathbb{R}^2$ with exponential critical growth. (English summary) Geometric properties for parabolic and elliptic PDE's, 251-267, Springer INdAM Ser., 2, Springer, Milan, 2013.

[23] N. S. Trudinger, On imbeddings into Orlicz spaces and some applications. J. Math. Mech. 17 1967 473-483.

[24] Y. Yang, Extremal functions for Moser-Trudinger inequalities on 2-dimensional compact Riemannian manifolds with boundary. (English summary) Internat. J. Math. 17 (2006), no. 3, 313-330.

[25] Y. Yang, Moser-Trudinger inequality for functions with mean value zero. (English summary) Nonlinear Anal. 66 (2007), no. 12, 2742-2755.