COMBINATORIAL LOCAL CONVEXITY IMPLIES CONVEXITY IN FINITE DIMENSIONAL CAT(0) CUBED COMPLEXES

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ABSTRACT. We give a proof of the following theorem, which is well-known among experts: A connected subcomplex W of a finite dimensional CAT(0) cubed complex X is convex if and only if $\mathrm{Lk}(v, W)$ is a full subcomplex of $\mathrm{Lk}(v, X)$ for every vertex v of W.

1. INTRODUCTION

The purpose of this note is to give a proof of the following theorem, which is well-known among experts.

Theorem 1.1. *Let X be a finite dimensional CAT(0) cubed complex and W a connected subcomplex of X. Then W is convex in X if and only if it satisfies the condition (CLC) below:*
(CLC) $\mathrm{Lk}(v, W)$ *is a full subcomplex of* $\mathrm{Lk}(v, X)$ *for every vertex v of W.*

Recall that a subcomplex K of a simplicial complex L is *full* if any simplex of L whose vertices are in K is in fact entirely contained in K. The condition (CLC) is nothing other than the definition for W to be "combinatorially locally convex" in X, in the sense of Haglund-Wise [5, Definition 2.9] (cf. Haglund [4, Definitions 2.8 and 2.9]). (Their terminology does not contain the adjective combinatorial.) In fact, they introduced the concept of a "combinatorial local isometry", and define W to be combinatorially locally convex in X if the inclusion map $j : W \to X$ is a combinatorial local isometry. As (implicitly) suggested in [10], Theorem 1.1 is an immediate consequence of [5, Lemma 2.11] concerning combinatorial local isometries from cube complexes to finite dimensional non-positively curved cube complexes.

In [5, Proof of Lemma 2.11] appealing to [1, Proposition II.4.14] (which is deduced from the classical Cartan-Hadamard theorem), it is implicitly assumed that a combinatorial local isometry is a local isometry in the usual sense (Definition 2.1(2)). On the other hand, Haglund writes in [4, the paragraph preceding Theorem 2.13] that in the finite dimensional case it can be checked that combinatorial local isometries are precisely local isometries of the ℓ_2 (Euclidean) metrics. Moreover, Petrunin notes

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in [10] that combinatorial local convexity implies local convexity and that this can be proved the same way as the flag condition (Gromov's link condition) for CAT(0) spaces. Thus Theorem 1.1 is established by [5, Lemma 2.11], though we could not find a reference that includes a proof of the implicit assertion.

The purpose of this note is to give a full proof of Theorem 1.1 by writing down a proof of the assertion (Theorem 2.2). Our proof totally depends on Bridson-Haefliger [1], and it may be regarded as a relative version of the proof of Gromov's link condition included in the book (see [1, Proofs of Theorems II.5.2 and II.5.20]).

The main bulk of this note was originally written as a part of [9]. After learning from [10] that Theorem 1.1 is well-known among experts (as we had expected) and that it is essentially contained in Haglund-Wise [5, Lemma 2.11], we decided to move that part of [9] into this separate note. We hope this note is of some use to those who are not so familiar with the relation between the two concepts concerning local convexity.

We note that Theorem 1.1 may be regarded as a Euclidean metric version of the combinatorial result by Haglund [4, Theorem 2.13], which shows that combinatorial convexity [4, Definition 2.9] is a local combinatorial property. However, Theorem 1.1 is weaker than [4, Theorem 2.13], in the sense that the former assumes finite dimensionality whereas the latter does not.

As is summarized in [8], local convexity implies (global) convexity in various settings, including the following:

- closed connected subsets in a Euclidean space (Nakajima [6] and Tietze [11]),
- closed connected subsets (whose diameter is less than $\pi/\sqrt{\kappa}$ when $\kappa > 0$) in a complete CAT(κ) space (Bux-Witzel [2, Theorems 1.6 and 1.10] and Ramos-Cuevas [8, Theorem 1.1]), and
- closed connected (by rectifiable arcs) subsets of proper Busemann spaces (Papadopoulos [7, Proposition 8.3.3]).

The following well-known fact is the simplest non-trivial example of such results.

- A local geodesic in a CAT(κ) space (of length less than $\pi/\sqrt{\kappa}$ when $\kappa > 0$) is a geodesic [1, Proposition I.1.4(2)].

This fact is repeatedly (though implicitly) used in this note.

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2. Basic definitions and outline of the proof of Theorem 1.1

We first recall basic facts concerning non-positively curved spaces following Bridson-Haefliger [1].

Let $X = (X, d)$ be a metric space. In this paper, we mean by a *geodesic* in X an isometric embedding $g : J \to X$ where J is a connected subset of $\mathbb{R}$. If J is a closed interval, we call g a *geodesic segment.* We do not distinguish between a geodesic and its image. X is a *geodesic space* if every pair of points can be joined by a geodesic in X. It is said to be *uniquely geodesic* if for every pair of points there is a unique geodesic joining them. For points a and b in a geodesic space X, we denote by $[a, b]$ a geodesic segment joining a and b. The symbols (a, b), $[a, b)$ and $(a, b]$ represent open or half-open geodesic segments, respectively. The distance $d(a, b)$ is equal to $\ell([a, b])$, the length of the geodesic segment $[a, b]$. Thus the geodesic space X is a *length space* in the sense that the distance between two points is the infimum over the lengths of rectifiable curves that join them [1, I.1.18 and I.3.1].

A geodesic space X is a *CAT(0) space* if any geodesic triangle is thinner than a comparison triangle in the Euclidean plane $\mathbb{E}^2$, that is, the distance between any points on a geodesic triangle is less than or equal to the corresponding points on a comparison triangle [1, Definition II.1.1]. A CAT(0) space is uniquely geodesic [1, Proposition II.1.4(1)]. A geodesic space X is said to be *non-positively curved* if it is locally a CAT(0) space, i.e., for every $x \in X$ there exists $r > 0$ such that the open r-ball $B_X(x, r) := \{y \in X \mid d(x, y) < r\}$ in X with center x, endowed with the induced metric, is a CAT(0) space [1, Definition II.1.2].

A *cubed complex* is a metric space $X = (X, d)$ obtained from a disjoint union of unit cubes $\hat{X} = \bigsqcup_{\lambda \in \Lambda}(I^{n_\lambda} \times \{\lambda\})$ by gluing their faces through isometries. To be precise, it is an M_κ-polyhedral complex with $\kappa = 0$ in the sense of [1, Definition I.7.37] that is made up of Euclidean unit cubes, i.e., the set Shapes(X) in the definition consists of Euclidean unit cubes. (See [1, Example (I.7.40)(4)].) The metric d on X is the length metric induced from the Euclidean metric of the unit cubes. See [1, I.7.38] for a precise definition. Every finite dimensional cubed complex is a complete geodesic space [1, Theorem in p.97 or I.7.33], where the *dimension* of the cubed complex is defined to be $\max\{n_\lambda\}$. Note that the restriction of the projection $p : \hat{X} \to X$ to $I^{n_\lambda} \times \{\lambda\}$ is not necessarily injective. Thus a cubed complex is not necessarily a *cube complex* in the sense of [4, 5]. However, the difference is not essential, because the cubical subdivision of a cubed complex (cf. [4, p.174]) is a cube complex, and the metrics of the cubed complex and its cubical decomposition (after rescaling) are identical (cf. [1, Lemma I.7.48]).

Two non-trivial geodesics issuing from a point $x \in X$ are said to define the *same direction* if the Alexandrov angle between them is zero. This defines an equivalence relation on the set of non-trivial geodesics issuing from x, and the Alexandrov angle induces a metric on the set of the equivalence classes. The resulting metric space is called the *space of directions* at x and denoted $S_x(X)$ [1, Definition II.3.18].

Suppose x is a vertex v of the cubed complex X. Then the space $S_v(X)$ is obtained by gluing the spaces $\{S_{v_\lambda}(I^{n_\lambda} \times \{\lambda\})\}_{v_\lambda \in p^{-1}(v)}$. Here $S_{v_\lambda}(I^{n_\lambda} \times \{\lambda\})$ is the space of directions in the cube $I^{n_\lambda} \times \{\lambda\}$ at the vertex v_λ; so it is an *all-right* *spherical simplex*, a geodesic simplex in the unit sphere $S^{n_\lambda - 1}$ all of whose edges have length $\pi/2$. Hence $S_v(X)$ has a structure of a finite dimensional *all-right spherical complex*, namely an M_κ-polyhedral complex with $\kappa = 1$ in the sense of [1, Definition I.7.37] that is made up of all-right spherical simplices, i.e., the set Shapes(X) in the definition consists of all-right spherical simplices. This complex is called the *geometric link* of v in X, and is denoted by $\mathrm{Lk}(v, X)$ [1, (I.7.38)]. It is endowed with the length metric $d_{\mathrm{Lk}(v,X)}$ induced from the spherical metrics of the all-right spherical simplices. Let $d^\pi_{\mathrm{Lk}(v,X)}$ be the metric defined by

$$d^\pi_{\mathrm{Lk}(v,X)}(u_1, u_2) := \min\{d_{\mathrm{Lk}(v,X)}(u_1, u_2), \pi\}.$$

Then the metric $d_{S_v(X)}$ on $S_v(X) = \mathrm{Lk}(v, X)$ is equal to the metric $d^\pi_{\mathrm{Lk}(v,X)}$ (see [1, the second sentence in p.191] or [9, Lemma 5.5]).

Definition 2.1. Let X be a uniquely geodesic space and W a subset of X.

(1) W is *convex* in X if, for any distinct points a and b in W, the unique geodesic segment $[a, b]$ in X is contained in W.

(2) W is *locally convex* in X if, for every $x \in W$, there is an $\epsilon > 0$ such that $W \cap B_X(x, \epsilon)$ is convex in X, where $B_X(x, \epsilon)$ is the open ϵ-ball in X with center x.

(3) Assume that X is a cubed complex and W is a subcomplex of X. Then W is *combinatorially locally convex* in X if it satisfies the the condition (CLC), i.e., $\mathrm{Lk}(v, W)$ is a full subcomplex of $\mathrm{Lk}(v, X)$ for every vertex v of W.

In the next section, we prove the following theorem.

Theorem 2.2. *Let X be a finite dimensional CAT(0) cubed complex and W a subcomplex of X. Then W is locally convex in X if and only if it is combinatorially locally convex in X.*

In the reminder of this section, we give a proof of Theorem 1.1 by using the above theorem and following [5, the proof of Lemma 2.11]. The starting point of the proof is the following version of the Cartan-Hadamard theorem.

Proposition 2.3. [1, Special case of Theorem II.4.1(2)] *Let X be a complete, connected, geodesic space. If X is non-positively curved, then the universal covering $\tilde{X}$ (with the induced length metric) is a CAT(0) space.*

See [1, Definition I.3.24] for the definition of the *induced length metric* on $\tilde{X}$. The Cartan-Hadamard theorem implies the following result [1, Proposition II.4.14], which plays an essential role in [5, Proof of Lemma 2.11] and so in the proof of Theorem 1.1.

Proposition 2.4. [1, Proposition II.4.14] *Let X and Y be a complete, connected metric space. Suppose that X is non-positively curved and that Y is locally a length space. If there is a map $f : Y \to X$ that is locally an isometric embedding, then Y is non-positively curved and:*

(1) *For every* $y_0 \in Y$*, the homomorphism* $f_* : \pi_1(Y, y_0) \to \pi_1(X, f(y_0))$ *induced by* f *is injective.*
(2) *Consider the universal coverings* $\tilde{X}$ *and* $\tilde{Y}$ *with their induced length metrics. Every continuous lifting* $\tilde{f} : \tilde{Y} \to \tilde{X}$ *of* f *is an isometric embedding.*

In the above proposition, $f : Y \to X$ being *locally an isometric embedding* means that, for every $y \in Y$, there is an $\epsilon > 0$ such that the restriction of f to the open ϵ-ball $B_Y(y, \epsilon)$ in Y is an isometry onto its image in X [1, the sentence preceding Proposition II.4.14].

We now give a proof of Theorem 1.1 following [5, Proof of Lemma 2.11] and assuming Theorem 2.2.

Proof of Theorem 1.1. Let X be a finite dimensional CAT(0) cubed complex and W a connected subcomplex of X. Suppose W is combinatorially locally convex. Then W is locally convex by Theorem 2.2.

Claim 2.5. *The inclusion map* $j : W \to X$*, regarded as a map between cubed complexes, is locally an isometric embedding, namely, for every* $x \in W$*, there is an* $\epsilon > 0$ *such that the restriction of* j *to the open* ϵ*-ball* $B_W(x, \epsilon)$ *in* W *(with respect to the metric* d_W *of the cubed complex* W*) is an isometry onto its image in the cubed complex* X*.*

Proof. Let $\epsilon > 0$ be such that $W \cap B_X(x, \epsilon)$ is convex in X. Then for any $a, b \in W \cap B_X(x, \epsilon)$, the geodesic $[a, b]$ in X is contained in $W \cap B_X(x, \epsilon)$. By the definitions of d_X and d_W as length metrics induced from the Euclidean metrics of the unit cubes, we see that $[a, b]$ is also a geodesic in W and $d_X(a, b) = d_W(a, b)$. Hence the restriction of $j : W \to X$ to the subspace $W \cap B_X(x, \epsilon) \subset W$ is an isometry onto its image $W \cap B_X(x, \epsilon) \subset X$. The above observation also implies that $W \cap B_X(x, \epsilon) \subset B_W(x, \epsilon)$. Since $B_W(x, \epsilon) \subset W \cap B_X(x, \epsilon)$ obviously holds, we have $W \cap B_X(x, \epsilon) = B_W(x, \epsilon)$. Hence, the restriction of $j : W \to X$ to the subspace $B_W(x, \epsilon) \subset W$ is an isometry onto its image in X. □

Since both X and W are complete [1, Theorem in p.97 or I.7.33] and since (W, d_W) is a length metric space, Claim 2.5 enables us to apply Proposition 2.4 ([1, Proposition II.4.14]) to $j : W \to X$, and so the following hold.

(0) W is non-positively curved.
(1) $j_* : \pi_1(W) \to \pi_1(X)$ is injective.
(2) Consider the universal coverings $\tilde{X}$ and $\tilde{W}$ with their induced length metrics. Every continuous lifting $\tilde{j} : \tilde{W} \to \tilde{X}$ of j is an isometric embedding.

Since X is a CAT(0) space, $\pi_1(X) = 1$ and so $\pi_1(W) = 1$ by the conclusion (1). Thus $W = \tilde{W}$ and it is a CAT(0) space by the conclusion (0) and the Cartan-Hadamard theorem (Proposition 2.3). Hence, by the conclusion (2), $j : W \to X$ is an isometric embedding of the cubed complex $W = \tilde{W}$ into the cubed complex

$X = \tilde{X}$. Thus, for any $a, b \in W$, the unique geodesic $[a, b]$ in the CAT(0) space W is also a geodesic in X. This means that $W = j(W)$ is convex in X, completing the proof of the if part.

The only if part immediately follows from the only if part of Theorem 2.2. □

Remark 2.6. (1) In [1, Proof of Proposition II.4.14], the proof of the assertion that Y is non-positively curved is rather involved, because it only assumes that the complete metric space Y is locally a length space. However, in our setting $Y = W$ is a connected subcomplex of the CAT(0) cubed complex which is combinatorially locally convex. So, the assertion in our setting is an immediate consequence of Gromov's link condition [1, Theorem II.5.20] (cf. Lemma 3.4(2)).

(2) If we appeal to the relatively new results by Bux-Witzel [2, Theorems 1.6 and 1.10] and Ramos-Cuevas [8, Theorem 1.1], which in particular imply that a closed connected subset of a complete CAT(0) space is convex if and only if it is locally convex, then Theorem 1.1 immediately follows from Claim 2.5.

3. Proof of Theorem 2.2

We begin by recalling basic properties of CAT(1) spaces. A metric space $L = (L, d)$ is a *CAT(1) space* if it is a geodesic space all of whose geodesic triangles of perimeter less than 2π are not thicker than its comparison triangle in the 2-sphere S^2 [1, Definition II.1.1].

Proposition 3.1. (1) ([1, Theorem II.5.4]) *Any CAT(1) space is uniquely π-geodesic, namely, for any points a and b of the space with $d(a, b) < \pi$, there is a unique geodesic $[a, b]$ joining a to b.*

(2) ([1, Theorem II.5.18]) *A finite dimensional all-right angled spherical complex is CAT(1) if and only if it is a flag complex.*

Recall that a *flag complex* is a simplicial complex in which every finite set of vertices that is pairwise joined by an edge spans a simplex.

Definition 3.2. ([1, Definition I.5.6]) For a metric space $Y = (Y, d_Y)$, the 0-*cone (or the Euclidean cone) $C_0(Y)$ over Y* is the metric space defined as follows. As a set $C_0(Y)$ is obtained from $[0, \infty) \times Y$ by collapsing $0 \times Y$ into a point. The equivalence class of (t, y) is denoted by ty, where the class of $(0, y)$ is denoted by 0 and is called the *cone point*. The distance $d(ty, t'y')$ between two points ty and $t'y'$ in $C_0(Y)$ is defined by the identity

$$d(ty, t'y')^2 = t^2 + t'^2 - 2tt' \cos(d_Y^{\pi}(y, y')),$$

where $d_Y^{\pi}(y, y') = \min\{d_Y(y, y'), \pi\}$.

For a vertex v in a cubed complex X, we denote the 0-cone $C_0(\mathrm{Lk}(v, X))$ by $T_v(X)$ and call it the *tangent cone* at v [1, Definition II.3.18].

We have the following fundamental relation between CAT(0) spaces and CAT(1) spaces, where the second statement (Gromov's link condition) is proved by using the first statement (Berestovskii's theorem).

Proposition 3.3. (1) (Berestovskii [1, Theorem II.3.14]) *Let $Y = (Y, d_Y)$ be a metric space. Then the 0-cone $C_0(Y)$ over Y is a CAT(0) space if and only if Y is a CAT(1) space.*

(2) (Gromov's link condition) [1, Theorem II.5.20] *A finite dimensional cubed complex X is non-positively curved if and only if, for every vertex $v \in X$, the geometric link $\mathrm{Lk}(v, X)$ is a CAT(1) space.*

The following lemma is a simple consequence of the above results.

Lemma 3.4. *Let X be a finite dimensional CAT(0) cubed complex and W a connected subcomplex of X. Then the following hold.*

(1) *For a vertex v of W, if $\mathrm{Lk}(v, W)$ is a full subcomplex of $\mathrm{Lk}(v, X)$ then the tangent cone $T_v(W)$ is a CAT(0) space.*

(2) *If $\mathrm{Lk}(v, W)$ is a full subcomplex of $\mathrm{Lk}(v, X)$ for every vertex v of W, then the cubed complex W is non-positively curved.*

Proof. (1) Since X is a CAT(0) cubed complex, $\mathrm{Lk}(v, X)$ is a flag complex by Proposition 3.3(2). If $\mathrm{Lk}(v, W)$ is a full subcomplex of $\mathrm{Lk}(v, X)$, then $\mathrm{Lk}(v, W)$ is also a flag complex. So, the all-right spherical complex $\mathrm{Lk}(v, W)$ is CAT(1) by Proposition 3.1(2). Hence, $T_v(W)$ is a CAT(0) space by Proposition 3.3(1).

(2) is proved by a similar argument by using Proposition 3.3(2) instead of Proposition 3.3(1) in the last step. □

Next, we prove the following key lemma for the proof of Theorem 2.2.

Lemma 3.5. *Let $L = (L, d)$ be a finite dimensional all-right spherical complex that is a flag complex, and let K be a subcomplex of L. Then the following conditions are equivalent.*

(1) *K is π-convex in L, namely, for any points a and b of K with $d(a, b) < \pi$, the unique geodesic segment $[a, b]$ in L is contained in K.*

(2) *K is a full subcomplex of L.*

Proof. We first prove that (1) implies (2). Suppose that K is not full in L. Then there is a simplex σ of $L \setminus K$ such that $\partial\sigma$ is contained in K. Pick a vertex v of σ, and let τ be the codimension 1 face of σ that does not contain the vertex v. Pick a point y in the interior of τ. Then $d(v, y) = \pi/2$ and the interior of the geodesic segment $[v, y]$ is contained in the interior of σ. Thus $[v, y]$ is not contained in K though both v and y are contained in K. Hence K is not π-convex.

We next prove that (2) implies (1). Suppose to the contrary that K is not π-convex though K is a full subcomplex of L. Then there is a geodesic segment $[a, b]$ in L of length $< \pi$ such that $a, b \in K$ but $[a, b] \not\subset K$. If necessary, by replacing

$[a,b]$ with a sub geodesic segment, we may assume $K \cap [a,b] = \{a,b\}$. Let σ be the simplex of L whose interior intersects the germ of $[a,b]$ at a. Then σ is not a simplex of K. Since K is a full subcomplex of L by the assumption, there is a vertex v of σ that is not contained in K. Let $\mathrm{St}(v,L)$ (resp. $\mathrm{st}(v,L)$) be the *closed star* (resp. *open star*) of v in L, i.e., the union of the simplices (resp. the interior of the simplices) of L that contain v. Note that $\mathrm{St}(v,L) = \mathrm{st}(v,L) \sqcup \mathrm{lk}(v,L)$, where $\mathrm{lk}(v,L)$ is the *simplicial link* of v in L, i.e., the union of the simplices τ of L such that $v \notin \tau$ and $\{v\} \cup \tau$ is contained in a simplex of L. Then $\mathrm{st}(v,L) \cap K = \emptyset$ and therefore there is a point $b' \in (a,b]$ such that $b' \in \mathrm{lk}(v,L)$ and $(a,b') \subset \mathrm{st}(v,L)$.

Case 1. $v \in (a,b')$. Then $d(v,a) = d(v,b') = \pi/2$ and hence $d(a,b) \geq d(a,b') = d(a,v) + d(v,b') = \pi$, a contradiction.

Case 2. $v \notin (a,b')$. We consider the "development" of $[a,b'] \subset \mathrm{St}(v,L)$ in the northern hemisphere S^2_+, the closed ball of radius $\pi/2$ centered at the north pole $N = (0,0,1)$ in S^2, that is defined as follows (cf. [1, Definition I.7.17]). Let $a = y_0, y_1, \cdots, y_n = b'$ be points lying in $[a,b']$ in this order, such that (y_{i-1}, y_i) is contained in the interior of a simplex $\sigma(i)$ of L for each i ($1 \leq i \leq n$). Note that $\sigma(i)$ contains v as a vertex. Let $\bar{y}_0 = (1,0,0), \bar{y}_1, \cdots, \bar{y}_n$ be the points in S^2_+ satisfying the following conditions.

(1) $d_{S^2}(N, \bar{y}_i) = d_{\sigma(i)}(v, y_i) = d(v, y_i)$ and $d_{S^2}(\bar{y}_{i-1}, \bar{y}_i) = d_{\sigma(i)}(y_{i-1}, y_i) = d(y_{i-1}, y_i)$ for each i.
(2) If $N, \bar{y}_{i-1}, \bar{y}_i$ are not aligned, the initial vectors of the geodesic segments $[N, \bar{y}_{i-1}]$ and $[N, \bar{y}_i]$ in S^2_+ occur in the order of a fixed orientation of S^2.

We call the union $\gamma := \cup_{i=1}^n [\bar{y}_{i-1}, \bar{y}_i] \subset S^2_+$ the *development* of $[a,b'] \subset \mathrm{St}(v,L)$ in S^2_+. It should be noted that $n \geq 2$ and $\bar{y}_1, \cdots, \bar{y}_{n-1}$ are contained in $\mathrm{int}\, S^2_+$.

Claim 3.6. *γ is a local geodesic in S^2.*

Proof. Though this is used without proof in [1, the 4th paragraph in the proof of Theorem II.5.18], we give a proof for completeness. If γ is not a local geodesic, then $\ell([\bar{y}_{i-1}, \bar{y}_i] \cup [\bar{y}_i, \bar{y}_{i+1}]) > \ell([\bar{y}_{i-1}, \bar{y}_{i+1}])$ for some i. Let $\bar{y}'_i$ be the intersection of the geodesic segment $[\bar{y}_{i-1}, \bar{y}_{i+1}]$ and the maximal geodesic segment in S^2_+ emanating from N and passing through $\bar{y}_i$. Let y'_i be the point in the maximal geodesic segment in $\sigma(i) \cap \sigma(i+1) \subset L$ emanating from v and passing through y_i, such that $d(v, y'_i) = d_{S^2}(N, \bar{y}'_i)$. Then we have the following isometries among spherical triangles.

$$\Delta(v, y_{i-1}, y'_i) \cong \Delta(N, \bar{y}_{i-1}, \bar{y}'_i), \quad \Delta(v, y'_i, y_{i+1}) \cong \Delta(N, \bar{y}'_i, \bar{y}_{i+1})$$

Hence the following hold.

$$\begin{aligned}
\ell([y_{i-1}, y'_i] \cup [y'_i, y_{i+1}]) &= \ell([\bar{y}_{i-1}, \bar{y}'_i] \cup [\bar{y}'_i, \bar{y}_{i+1}]) \\
&= \ell([\bar{y}_{i-1}, \bar{y}_{i+1}]) \\
&< \ell([\bar{y}_{i-1}, \bar{y}_i] \cup [\bar{y}_i, \bar{y}_{i+1}]) \\
&= \ell([y_{i-1}, y_i] \cup [y_i, y_{i+1}]) = \ell([y_{i-1}, y_{i+1}])
\end{aligned}$$

This contradicts the fact that $[y_{i-1}, y_{i+1}](\subset [a, b'] \subset [a, b])$ is a geodesic. $\square$

Since γ is a local geodesic with length $\ell(\gamma) < \pi$, it is a geodesic in S^2_+ by [1, Proposition II.1.4(2)]. Since $y_n = b' \in \mathrm{lk}(v, L)$, we see $d(v, y_n) = \pi/2$ and so $\bar{y}_n \in \partial S^2_+$. Thus the endpoints $\bar{y}_0$ and $\bar{y}_n$ of the geodesic $\gamma \subset S^2_+$ are contained in ∂S^2_+. Since $\ell(\gamma) < \pi$, this implies $\gamma \subset \partial S^2_+$. This contradicts the fact that $\bar{y}_1, \cdots, \bar{y}_{n-1}$ are contained in $\mathrm{int}\, S^2_+$. This completes the proof of Lemma 3.5. $\square$

In addition to Lemma 3.5, we need Lemma 3.8 below which gives relative versions of two results included in [1] concerning the local shape of polyhedral complexes.

Notation 3.7. For a vertex v of a subcomplex W of a cubed complex X, the symbol $j : T_v(W) \to T_v(X)$ denotes the natural injective map from the tangent cone $T_v(W)$ of the cubed complex W into the tangent cone $T_v(X)$ of the cubed complex X. Note that j is not necessarily an isometric embedding.

Lemma 3.8. *Let X be a finite dimensional cubed complex and W a subcomplex of X. Then the following hold.*

(1) (Relative version of [1, Theorem I.7.39]) *Let v be a vertex of W. Then there is a natural isometry φ from the open ball $B_X(v, 1/2)$ in X onto the open ball $B_{T_v(X)}(0, 1/2)$ in the tangent cone $T_v(X)$ that carries $W \cap B_X(v, 1/2)$ onto $j(T_v(W)) \cap B_{T_v(X)}(0, 1/2)$.*

(2) (Relative version of [1, Lemma I.7.56]) *Let x and y be points of W contained in the same open cell of W. Then, for sufficiently small $\epsilon > 0$, there exists a natural isometry between the open balls $B_X(x, \epsilon)$ and $B_X(y, \epsilon)$ in X that carries $W \cap B_X(x, \epsilon)$ onto $W \cap B_X(y, \epsilon)$.*

Proof. (1) By [1, Theorem I.7.39], there is a natural isometry from $B_X(v, 1/2)$ onto $B_{T_v(X)}(0, 1/2)$. (The radius $1/2$ is the half of the length 1 of the unit interval I, and it corresponds to $\varepsilon(x)/2$ in [1, Theorem I.7.39].) The isometry is defined as follows (see [1, the first paragraph in the proof of Theorem I.7.16 in p.104]). If $x \in B_X(v, 1/2)$ then there is a direction $u \in \mathrm{Lk}(v, X)$ such that x is a distance $t < 1/2$ along the geodesic issuing from v in the direction u. (Here u is uniquely determined by x except when $x = v$, i.e., $t = 0$.) Then $x \in B_X(v, 1/2)$ is mapped to the point $tu \in B_{T_v(X)}(0, 1/2)$. By this definition of the isometry, we see that it carries $W \cap B_X(v, 1/2)$ onto $j(T_v(W)) \cap B_{T_v(X)}(0, 1/2)$.

(2) By [1, Lemma I.7.56], there is a natural isometry from $B_X(x, \epsilon)$ onto $B_X(y, \epsilon)$ that restricts to an isometry from $C \cap B_X(x, \epsilon)$ onto $C \cap B_X(y, \epsilon)$ for every closed cell C of X containing x and y. Obviously the isometry carries $W \cap B_X(x, \epsilon)$ onto $W \cap B_X(y, \epsilon)$. $\square$

We now give a proof of the main Theorem 2.2.

Proof of Theorem 2.2. Let X be a finite dimensional CAT(0) cubed complex and W a subcomplex of X. Assume that W is combinatorially locally convex in X, i.e.,

$\mathrm{Lk}(v, W)$ is a full subcomplex of $\mathrm{Lk}(v, X)$ for every vertex v of W. Then we have the following claim.

Claim 3.9. *For any vertex v of W, the map $j : T_v(W) \to T_v(X)$ is an isometric embedding, and $j(T_v(W))$ is convex in $T_v(X)$.*

Proof. Let v be a vertex of W. Then, by the assumption and Lemma 3.5, $\mathrm{Lk}(v, W)$ is π-convex in $\mathrm{Lk}(v, X)$. This implies that the distance function $d^{\pi}_{\mathrm{Lk}(v,W)}$ on $\mathrm{Lk}(v, W)$ is equal to the restriction of the distance function $d^{\pi}_{\mathrm{Lk}(v,X)}$ on $\mathrm{Lk}(v, X)$ to the subspace $\mathrm{Lk}(v, W)$. Hence $j : T_v(W) \to T_v(X)$ is an isometric embedding. On the other hand, $T_v(W)$ is a CAT(0) space by Lemma 3.4(1). Hence, any two points of $T_v(W)$ are joined by a unique geodesic in the metric space $T_v(W)$. Its image in $T_v(X)$ is also a geodesic in the metric space $T_v(X)$, because $j : T_v(W) \to T_v(X)$ is an isometric embedding. Hence $j(T_v(W))$ is convex in $T_v(X)$ as desired. □

Now let x be an arbitrary point in W. Pick a vertex v of the open cell of W that contains x. Then, by Lemma 3.8(2), we can find a small real $\epsilon > 0$ and $x' \in B_X(v, 1/2)$ with $B_X(x', \epsilon) \subset B_X(v, 1/2)$, such that $(B_X(x, \epsilon), W \cap B_X(x, \epsilon))$ is isometric to $(B_X(x', \epsilon), W \cap B_X(x', \epsilon))$. Recall the following isometry given by Lemma 3.8(1).

$$\varphi : (B_X(v, 1/2), W \cap B_X(v, 1/2)) \to (B_{T_v(X)}(0, 1/2), j(T_v(W)) \cap B_{T_v(X)}(0, 1/2))$$

Since $B_X(x', \epsilon) \subset B_X(v, 1/2)$, we have the following identities.

$$\varphi(B_X(x', \epsilon)) = B_{T_v(X)}(\varphi(x'), \epsilon),$$
$$\varphi(W \cap B_X(x', \epsilon)) = j(T_v(W)) \cap B_{T_v(X)}(\varphi(x'), \epsilon).$$

Since $j(T_v(W))$ is convex in $T_v(X)$ by Claim 3.9 and since $B_{T_v(X)}(\varphi(x'), \epsilon)$ is convex in the CAT(0) space $T_v(X)$ by [1, Proposition II.1.4(3)], these identities imply that $\varphi(W \cap B_X(x', \epsilon))$ is convex in the convex subset $B_{T_v(X)}(\varphi(x'), \epsilon)$ of $T_v(X)$. Since we have the isometries

$$\begin{aligned}(B_X(x, \epsilon), W \cap B_X(x, \epsilon)) &\cong (B_X(x', \epsilon), W \cap B_X(x', \epsilon)) \\ &\cong (\varphi(B_X(x', \epsilon)), \varphi(W \cap B_X(x', \epsilon))),\end{aligned}$$

this in turn implies that $W \cap B_X(x, \epsilon)$ is convex in the convex subset $B_X(x, \epsilon)$ of X. Hence $W \cap B_X(x, \epsilon)$ is convex in X, completing the proof of the only if part of Theorem 2.2.

Though the if part of Theorem 2.2 may be trivial, we include a proof for completeness. Suppose that $\mathrm{Lk}(v, W)$ is not a full subcomplex of $\mathrm{Lk}(v, X)$. Then $\mathrm{Lk}(v, W)$ is not π-convex by Lemma 3.5, and so there is a geodesic segment $[a, b]$ in $\mathrm{Lk}(v, X)$ such that $[a, b] \cap \mathrm{Lk}(v, W) = \{a, b\}$. Pick a small $t > 0$ so that the geodesic $[ta, tb]$ in $T_v(X)$ is contained in the open ball $B_{T_v(X)}(0, 1/2)$. (In fact, any positive $t < 1/2$

works.) Since the geodesic $[ta, tb]$ intersects $j(T_v(W))$ only at the endpoints, the inverse image of $[ta, tb]$ by the isometry φ in Lemma 3.8(1) is a geodesic in $B_X(v, 1/2)$ that intersects W only at the endpoints. Hence W is not locally convex. □

References

[1] M. Bridson and A. Heafliger, *Metric spaces of non-positive curvature*, Springer-Verlag (1999).

[2] K.-W. Bux and S. Witzel *Local convexity in CAT(κ)-spaces*, arXiv:1211.1871 [math.MG].

[3] M. Gromov, *CAT(κ)-spaces: construction and concentration*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **280** (2001), Geom. i Topol. **7**, 100–140, 299–300; reprinted in J. Math. Sci. (N.Y.) **119** (2004), no. 2, 178–200.

[4] F. Haglund, *Finite index subgroups of graph products*, Geom. Dedicata **135** (2008), 167–209.

[5] F. Haglund and D. Wise, *Special cube complexes*, Geom. Funct. Anal. **17** (2008), no. 5, 1551–1620.

[6] S. Nakajima, *Über konvexe Kurven und Flächen*, Tôhoku Math. J., **29** (1928), 227–230.

[7] A. Papadopoulos, *Metric spaces, convexity and nonpositive curvature*, IRMA Lectures in Mathematics and Theoretical Physics, **6**. European Mathematical Society (EMS), Zürich, 2005. xii+287 pp.

[8] C. Ramos-Cuevas, *Convexity is a local property in* CAT(κ) *spaces*, Mexican mathematicians abroad: recent contributions, 189–196, Contemp. Math., **657**, Aportaciones Mat., Amer. Math. Soc., Providence, RI, 2016.

[9] S. Sakai and M. Sakuma *Two-parabolic-generator subgroups of hyperbolic* 3*-manifold groups*, preprint.

[10] StackExchanges, *Convex subcomplexes of CAT(0) cubical complexes*, `https://mathoverflow.net/questions/194235/convex-subcomplexes-of-cat0-cubical-complexes`

[11] H. Tietze, *Über Konvexheit im kleinen und im großen und über gewisse den Punkten einer Menge zugeordnete Dimensionaszahlen*, Math. Z. **28** (1928), 679–707.

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