Whitehead aspherical conjecture via ribbon sphere-link

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ABSTRACT

Whitehead aspherical conjecture says that every connected subcomplex of every aspherical 2-complex is aspherical. For every contractible finite 2-complex, an argument on ribbon sphere-links allows us to confirm that the conjecture is true. In this paper, by generalizing this argument, this conjecture is confirmed to be true for every aspherical 2-complex.

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1. Introduction

A 2-complex is a finite or countably-infinite CW 2-complex, which is constructed from a finite or countably-infinite CW 1-complex homeomorphic to a simplicial 1-complex (namely, a graph) by attaching a finite or countably-infinite family of 2-cells with attaching maps. A 2-complex is homotopy equivalent to a simplicial 2-complex by taking a simplicial 1-complex homeomorphic to the CW 1-complex and a simplicial approximation of every attaching map of a 2-cell. By this homotopy equivalence, every subcomplex of a 2-complex is also homotopy equivalent to a simplicial subcomplex of the simplicial 2-complex (cf. Spanier Spanier for a general reference). The *2-complex* of a group presentation $< x_1, x_2, \dots, x_n, \dots \,|\, r_1, r_2, \dots, r_m, \cdots >$ is the 2-complex obtained from the 1-complex whose fundamental group is isomorphic to the free group with a basis of the generationg set $x_1, x_2, \dots, x_n, \dots$ and the attaching 2-cells given by the relators $r_1, r_2, \dots, r_m, \dots$. Up to homotopy equivalences, every connected 2-complex can be considered as the 2-complex of a group presentation $< x_1, x_2, \ldots, x_n, \ldots \,|\, r_1, r_2, \ldots, r_m, \cdots >$ and every connected subcomplex of it is the 2-complex of the group presentation given by a sub presentation $< x_{i_1}, x_{i_2}, \ldots, x_{i_s}, \ldots \,|\, r_{j_1}, r_{j_2}, \ldots, r_{j_t}, \cdots >$. A path-connected space X is *aspherical* if the universal cover $\tilde{X}$ of X is contractible (i.e., homotopy equivalent to a point). For a connected 2-complex P, it is equivalent to saying that the second homotopty group $\pi_2(P, v) = 0$. The Whitehead asphericity conjecture is the following conjecture (see ([2, 14]).

Whitehead Aspherical Conjecture. Every connected subcomplex of any aspherical 2-complex is aspherical.

The purpose of this paper is to claim that this conjecture is yes. That is,

Theorem 1.1. Whitehead Aspherical Conjecture is true.

In [9], it is shown that every connected subcomplex of any finite contractible 2-complex is aspherical by using some properties of a ribbon sphere-link in the 4-sphere, which is a partial affirmartive solution of this conjecture. The proof of Theorem 1.1 is done by a generalization of this method.

The author found on the Internet the preprint of E. Pasku [12] reporting the same result which appears obtained by a purely group theoretic argument, much different from the present argument.

The proof is organised as follows. In Section 2, the conjecture for every connected subcomplex of any aspherical 2-complex is reduced to the conjecture for every finite connected subcomplex of any locally finite contractible 2-complex. In Section 3, base changes on an infinite rank free abelian group and an infinite rank free group are observed. In Section 4, every locally finite, infinite group presentation of the trivial group is realized as a locally linked, infinite ribbon link in the 4-space such that the free faudamental group is the free group with the generating set as a basis and and the relator set as a meridian system of the ribbon sphere-link. In Section 5, the conjecture for every finite connected subcomplex of any locally finite contractible 2-complex is confirmed to be true.

2. Reducing to the conjecture for a finite subcomplex

In this section, it is explained that Theorem 1.1 (Whitehead Asphericity Conjecture) is obtained from the following theorem.

Theorem 2.1. Every connected finite subcomplex of any locally finite contractible 2-complex is aspherical.

For this reduction, the following lemma is used.

Lemma 2.2. Every connected finite subcomplex of an infinite connected 2-complex P is a subcomplex of a locally finite connected 2-complex P' homotopy equivalent to P.

Proof of Lemma 2.2. Let P be a connected infinite 2-complex, and P_0 a connected finite subcomplex of P. Let

$$P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n \subset \dots$$

be a sequence of connected finite subcomplexes $P_n\,(n = 0, 1, 2, \dots, n, \dots)$ of P such that $P = \cup_{n=0}^{+\infty} P_n$. Let $P_n = P_{n-1} \cup J_n$ for a subcomplex J_n of P_n with $\gamma_n = P_{n-1} \cap J_n$ a graph for all n. Triangulate the rectangle $a \times [0, 1]$ for every 1-simplex a of γ_n by introducing a diagonal and regard the product $\gamma_n \times [0, 1]$ as a 2-complex. To construct a desired 2-complex P', make the connected finite subcomplexes $J_n\,(n = 1, 2, 3, \dots, n, \dots)$ disjoint. Let $P'_n = P_{n-1} \cup \gamma_n \times [0, 1]$ be the 2-complex obtained from the subcomplexes P_{n-1} and $gamma_n \times [0, 1]$ by identifying $\gamma_n(\subset P_{n-1})$ with $\gamma_n \times 0$ and $\gamma_n \times 1$ with $\gamma_n(\subset J_n)$ in canonical ways. The sequence

$$P_0 = P'_0 \subset P'_1 \subset P'_2 \subset \cdots \subset P'_n \subset \dots$$

of connected finite subcomplexes $P'_n\,(n = 0, 1, 2, \dots, n, \dots)$ is obtained. By construction, $P' = \cup_{n=0}^{\infty} P'_n$ is a connected locally finite 2-complex containing P_0 as a subcomplex and homotopy equivalent to P. □

By using Lemma 2.2, Theorem 1.1 is obtained from Theorem 2.1 as follows.

2.3: Proof of Theorem 1.1 assuming Theorem 2.1. Since the universal cover $\tilde{P}$ of an aspherical 2-complex P is a contractible 2-complex, every subcomplex Q of P lifts to a subcomplex $\tilde{Q}$ of $\tilde{P}$, and the second homotopy group is independent of a covering by the lifting property (cf. [13]), Theorem 1.1 will be obtained from the following assertion.

(2.3.1) Every connected subcomplex of a contractible 2-complex is aspherical.

Because the topology of the polyhedron $|P|$ of an infinite simplicial 2-complex P is the topology cohertent with the simplexes of P (see [13, p.111]), the image $f(S^2)$ of the 2-sphere S^2 by any map $f : S^2 \to |P|$ is in the polyhedron $|P^f|$ of a finite connected subcomplex P^f of P. Thus, the assertion (2.3.1) is obtained from the following assertion.

(2.4.1) Every connected finite subcomplex of a contractible 2-complex is aspherical.

By lemma 2.2, every connected finite subcomplex of a contractible 2-complex is a connected finite subcomplex of a locally finite contractible 2-complex. Hence the assertion (2.4.1) is obtained from Theorem 2.1. Thus, Theorem 1.1 is obtained from Theorem 2.1. □

3. Base changes on an infinite rank free abelian group and an infinite rank free group

A *base change* of a free group $\mathbf{F}$ on a basis $x_i\,(i = 1, 2, \dots, n, \dots)$ is a consequence of a finite number of the following operations, called *Nielsen transformations* (see [11]):

(1) Exchange two of $x_i\,(i = 1, 2, \dots, n, \dots)$,
(2) Replace an x_i by x_i^{-1},
(3) Replace an x_i by $x_i x_j\,(i \neq j)$.

A *base change* of a free abelian group $\mathbf{A}$ on a basis $a_i\,(i = 1, 2, \dots, n, \dots)$ is a consequence of a finite number of the following operations:

(1) Exchange two of $a_i\,(i = 1, 2, \dots, n, \dots)$,
(2) Replace an a_i by $-a_i$,
(3) Replace an a_i by $a_i + a_j\,(i \neq j)$.

The following lemma is well-known for a finite rank free abeliang group $\mathbf{A}$.

Lemma 3.1. Let $\mathbf{A}$ be a free abelian group with a countable basis $a_i\,(i = 1, 2, \dots, n, \dots)$. Let $b_i\,(i = 1, 2, \dots, n, \dots)$ be a countable basis of $\mathbf{A}$ such that any row or colum vector of the base change matrix C given by

$$(b_1 b_2 \dots b_n \dots) = (a_1 a_2 \dots a_n \dots)C$$

has only a finite number of non-zero entries. Then for every positive integer m, there is a base change of $\mathbf{A}$ on $a_i\,(i = 1, 2, \dots, n, \dots)$ such that C is equal to the block sum $E_m \oplus C'$ for the unit matrix E_m of size m and a matric C'.

Proof of Lemma 3.1. For every $i\,(i = 1, 2, \dots, n, \dots)$, let

$$b_i = c_{i1}a_1 + c_{i2}a_2 + \dots c_{in}a_n + \dots$$

be a linear combination with c_{ij} integers which are 0 except for a finite number of $j\,(j = 1, 2, \dots, n, \dots)$. Note that for every i, the non-zero integer system of $c_{i1}, c_{i2}, \dots, c_{in}, \dots$ is a coprime integer system. By a base change, assume that c_{11} is a smallest positive integer in the integers $|c_{1j}|$ (except for 0). Write $c_{1j} = \tilde{c}_{1j}c_{11} + d_{1j}$ for $0 \le d_{1j} < c_{11}$. By a base change on $a_i\,(i = 1, 2, \dots, n, \dots)$, assume that

$$b_1 = c_{11}a_1 + d_{12}a_2 + \dots d_{1n}a_n + \dots$$

By continuing this process, it can be assumed that $b_1 = a_1$. Next, consider the linear combination

$$b_2 = c_{21}a_1 + c_{22}a_2 + \dots c_{2n}a_n + \dots.$$

Note that for every $i \ge 2$, the non-zero integer system of $c_{22}, c_{23}, \dots, c_{2n}, \dots$ is coprime. By a base change on $a_i\,(i = 2, 3, \dots, n, \dots)$, it can be assumed that that $b_2 = c_{21}a_1 + a_2$. By an inductive argument, it can be assumed that

$$b_i = c_{i1}a_1 + c_{i2}a_2 + \dots c_{i\,i-1}a_{i-1} + a_i\,(i = 3, 4, \dots, m).$$

Let $m^+ \ge m$ be an integer such that for every $i > m^+$, $c_{ij} = 0\,(j = 1, 2, \dots, m)$. By continuing the inductive argument, it can be assumed that

$$b_i = c_{i1}a_1 + c_{i2}a_2 + \dots c_{i\,i-1}a_{i-1} + a_i\,(i = m + 1, m + 2, \dots, m^+).$$

By a base change replacing a_i to $a_i - c_{i1}a_1 - c_{i2}a_2 - \dots - c_{i\,i-1}a_{i-1}\,(i = 2, 3, \dots, m^+)$, it is obtained that

$$b_i = a_i\,(1 \le i \le m^+), \quad c_{ij} = 0\,(i > m^+,\, 1 \le j \le m)$$

This completes the proof of Lemma 3.1. □

4. A locally finite 2-complex and an infinite ribbon sphere-llink

Let X be an open connected oriented smooth 4D manifold. A countably infinite family of disjoint compact sets $X_i\,(i = 1, 2, \dots, n, \dots)$ in X is *descrete* if the set $\{x_i | \, i = 1, 2, \dots, n, \dots\}$ made from any one point $x_i \in X_i$ for every i is discrete in X. A *sphere-link*, also called an S^2-*link* in X is the union L of a finite or countably infinite discrete family of disjoint 2-spheres smoothly embedded in X. An S^2-link in X is *trivial* if it bounds a discrete family of mutually disjoint 3-balls smoothly embedded in X, and *ribbon* if it is obtained from a trivial S^2-link O by surgery along a discrete family of disjoint 1-handles embedded in X. An S^2-*link* L in X is *finite* if the number of the components of L is finite Otherwise, L is *infinite*.

Let $\mathbf{R}^4$ be the 4-space. Let $\mathbf{H}^4 = \{(x, y, z, w) | -\infty < x, y, z < +\infty, 0 \le w\}$ be the upper-half 4-space of $\mathbf{R}^4$ with boundary $\partial\mathbf{H}^4 = \{(x, y, z, 0) | -\infty < x, y, z < +\infty\}$ identifying the 3-space $\mathbf{R}^3 = \{(x, y, z) | -\infty < x, y, z < +\infty\}$.

For two oriented open 4D manifolds X and Y, assume that there are smooth embeddings $i_X : \mathbf{H}^4 \to X$ and $i_Y : \mathbf{H}^4 \to Y$ such that $X' = \mathrm{cl}(X \setminus i_X(\mathbf{H}^4))$ and $Y' = \mathrm{cl}(Y \setminus i_Y(\mathbf{H}^4))$ are oriented 4D manifolds with boundaries $\partial X' = i_X(\partial \mathbf{H}^4)$ and $\partial Y' = i_Y(\partial \mathbf{H}^4)$ diffeomorphic to $\mathbf{R}^3$, respectively. The oriented open 4D manifold obtained from X' and Y' by pasting $\partial X'$ and $\partial Y'$ with an orientation-reversing diffeomorphism is called an $\mathbf{R}^3$-*connected sum* of X and Y and denoted by $X\#_{\mathbf{R}^3}Y$. The *open 4D handlebody*

$$Y^O = \mathbf{R}^4 \#_{i=1}^{+\infty} S^1 \times S^3_i$$

with a discrete family of connected summands $S^1 \times S^3_i\,(i = 1, 2, \ldots, n, \ldots)$ has an important role of this paper.

Lemma 4.1. Assume that the 2-complex of a group presentation

$$< x_1, x_2, \ldots x_n, \ldots \,|\, r_1, r_2, \ldots, r_m, \cdots >$$

is a locally finite contractible 2-complex (in other words, every generator x_i appears only in a finite number of the relators $r_1, r_2, \ldots, r_m, \ldots$). Then there is a ribbon S^2-link L with components $K_i\,(i = 1, 2, \ldots, n, \ldots)$ in $\mathbf{R}^4$ such that

- the fundamental group $\pi_1(\mathbf{R}^4 \setminus L, v)$ is isomorphic to the free group with basis $x_i\,(i = 1, 2, \ldots, n, \ldots)$ by an isomorphism sending a meridian system of $K_i\,(i = 1, 2, \ldots, n, \ldots)$ to the relator system $r_i\,(i = 1, 2, \ldots, n, \ldots)$, and
- an $\mathbf{R}^3$-connected sum $Y\#_{\mathbf{R}^3}X$ for the 4D manifold Y obtained from $\mathbf{R}^4$ by surgery along L and a contractible smooth open 4D manifold X is diffeomorphic to the open 4D handlebody Y^O.

The ribbon S^2-link L in $\mathbf{R}^4$ is referred to as a *ribbon S^2-link associated with the group presentation* $< x_1, x_2, \ldots x_n, \ldots \,|\, r_1, r_2, \ldots, r_m, \cdots >$.

Proof of Lemma 4.1. Since the 2-complex of the group presentation is a locally finite contractible 2-complex, every generator x_i appears in only a finite number of the relators $r_1, r_2, \ldots, r_m, \ldots$ and the inclusion homomorphism

$$< r_1, r_2, \ldots, r_m, \cdots > \to < x_1, x_2, \ldots x_n, \cdots >$$

induces an isomorphism on the abelianized groups which are free abelian groups. By a base change on $x_i\,(i = 1, 2, \ldots, n, \ldots)$, it is assumed from Lemma 3.1 that the word r_1 is equal to the letter x_1 in the abelianized group of the free group $< x_1, x_2, \ldots x_n, \cdots >$. Let m be a positive integer such that every letter x_j contained in the word r_1 belongs to the letters $x_i\,(i = 1, 2, \ldots, m)$. Further, by a base change on $x_i\,(i = 1, 2, \ldots, n, \ldots)$, it is assumed from Lemma 3.1 that the words $r_i\,(i = 1, 2, \dots, m)$ are equal to the letters $x_i\,(i = 1, 2, \dots, m)$, respectively in the abelianized group of the free group $< x_1, x_2, \dots x_n, \cdots >$. In the open 4D handlebody $Y^O = \mathbf{R}^4 \#_{i=1}^{+\infty} S^1 \times S^3_i$, let $x_i = [k^O_i]\,(i = 1, 2, \dots, n, \dots)$ be a basis of the free group $\pi_1(Y^O, v)$ represented by the loop $k^O_i = S^1 \times \mathbf{1}_i$, and $r_i = [k_i]\,(i = 1, 2, \dots, n, \dots)$ an element system in $\pi_1(Y^O, v)$ represented by γ for every i. By assumption, the loop k_i meets transversely $1 \times S^3_i$ with the intersection number $+1$. Every loop $k_j\,(j \neq i)$ does not meet $1 \times S^3_i$ except for a finite number of j and when it meets, it meets transversely with the intersection number 0. Let X be the smooth open 4D manifold obtained from Y^O by surgery along the loops $k_i\,(i = 1, 2, \dots, n, \dots)$ using a normal D^3-bundle $k_i \times D^3$ of k_i in Y^O, which are changed into normal D^2-bundles $D_i \times S^2\,(i = 1, 2, \dots, n, \dots)$ of the S^2-link $L = \cup_{i=1}^{+\infty} K_i$ with $K_1 = 0_i \times S^2$ in X.

(4.1.1) The open 4D manifold X is contractible.

Proceed with the proof by assuming (4.1.1). By an argument of [8, Lemma 3.4], the 2-sphere K_1 is isotopic to a ribbon S^2-knot in X obtained from a finite trivial S^2-link O_1 split from L by surgery along a finite number of disjoint 1-handles whose core arcs possibly pass through only the meridians of the S^2-knots $K_j\,((j = 1, 2, \dots, m)$. Every the 2-sphere K_i has a similar situation. This means that the S^2-link L is a ribbon S^2-link in X. Consider X as an $\mathbf{R}^3$-connected sum $X \#_{\mathbf{R}^3} \mathbf{R}^4$ by taking a smooth embedding $i_X : \mathbf{H}^4 \to X$ for the upper-half 4-space $\mathbf{H}^4$. Then the ribbon S^2-link L can be moved into the connected summand $\mathbf{R}^4$ of the $\mathbf{R}^3$-connected sum $X \#_{\mathbf{R}^3} \mathbf{R}^4$, since L is obtained from a trivial S^2-link which is movable into $\mathbf{R}^4$ by surgery along a discrete family of disjoint 1-handles which is also movable into the connected summand $\mathbf{R}^4$. Let Y be the open 4D manifold obtained from $\mathbf{R}^4$ by surgery along L. Then an $\mathbf{R}^3$-connected sum $Y \#_{\mathbf{R}^3} X$ is diffeomorphic to Y^O. This completes the proof of Lemma 4.1. □

The proof of (4.1.1) is done as follows.

Proof of (4.1.1). By van Kampen theorem, X is simply connected because the loops $k_i\,(i = 1, 2, \dots, n, \dots)$ normally generate the fundamental group $\pi_1(Y^O, v)$. Since X is an open 4D manifold, to know that $X$is contractible, it is enough to show that $H_q(X; \mathbf{Z}) = 0\,(q = 2, 3)$. By the excision isomorphism

$$H_q(Y^O, k_* \times D^3; \mathbf{Z}) \cong H_q(X, D_* \times S^2; \mathbf{Z}),$$

we have $H_3(X, D_* \times S^2; \mathbf{Z}) = 0$, so that $H_3(X; \mathbf{Z}) = 0$. Note that the Nielsen transformations are realized by orientation-preserving diffeomorphisms of Y^O. Then by Lemma 3.1, for each loop k_i in Y^D, there is a 3-sphere S^3_i in Y^D meeting k_i with intersection number $+1$ and meeting only finitely many loops $k_j\,(j\neq i)$ with intersection number 0. Thus, the S^2-knot K_i bounds in X a once-punctured 3D manifold of a 3D closed handlebody $S^3\#sS^1\times S^2$ for some s not meeting the other S^2-knots $K_j\,(j\neq i)$. This means that the inclusion homomorphism $H_2(D_*\times S^2;\mathbf{Z})\to H_2(X;\mathbf{Z})$ is the zero map. Since $H_2(X, D_*\times S^2;\mathbf{Z})\cong H_2(Y^O, k_*\times D^3;\mathbf{Z}=0$, we have $H_2(X;\mathbf{Z})=0$. This completes the proof of (4.1.1). □

5. Proof of Theorem 2.1

Let α be the reflection in $\mathbf{R}^4$ sending (x,y,z,w) to $(x,y,z,-w)$. The image $\alpha(H^4)$ of the upper-half 4-space H^4 by α is given by the lower-half 4-space $\{(x,y,z,w)|\,0<x,y,z<+\infty, w\leq 0\}$. A *disk-link* L^D in H^4 is a (finite or countably infinite) discrete family of disjoint disks smoothly and properly embedded in $\mathbf{H}^4$. The disk-link L^D in H^4 is *trivial* if it is obtained from a discrete family of disjoint disks in $\mathbf{R}^3$ by pushing the interiors into the interior of $\mathbf{H}^4$, and *ribbon* if it is obtained from a trivial disk-link in $\mathbf{H}^4$ and a discrete family of spanning bands in $\mathbf{R}^3$ by pushing the interior of the disk family which is the union of the trivial disk-link and the spanning bands into the interior of $\mathbf{H}^4$. The *closed exterior* of a ribbon disk-link L^D in $\mathbf{H}^4$. is the 4D manifold $E(L^D)=\mathrm{cl}(\mathbf{H}^4\setminus N(L^D))$ for a regular neighborhood of L^D in $\mathbf{H}^4$. The following lemma is analogous to [9, Lemma 4.1], but for completeness the proof for an infinite ribbon disk-link L^D is given.

Lemma 5.1. The closed exterior $E(L^D)$ of every ribbon disk-link L^D in $\mathbf{H}^4$ has a handle decomposition consisting of $\mathbf{H}^4$, a discrete family of disjoint 1-handles and a discrete family of disjoint 2-handles. In particular, the closed exterior $E(L^D)$ is homotopy equivalent to a connected 2-complex.

Proof of Lemma 5.1. The ribbon disk-link L^D in $\mathbf{H}^4$ is given by the union

$$\cup_{i=1}^{+\infty} d_i \cup_{j=1}^{+\infty} b_j$$

for a trivial proper disk system $d_i\,(i=1,2,\dots,n,\dots)$ in $\mathbf{H}^4$ and a band system $b_j\,(j=1,2,\dots,n,\dots)$ lifting the band system $b^0_j\,(j=1,2,\dots,n,\dots)$ in $\partial\mathbf{H}^4=\mathbf{R}^3$. Let $h_j\,(j=1,2,\dots,n,\dots)$ be the 1-handle system obtained as the lifting trace of the band system $b^0_j\,(j=1,2,\dots,n,\dots)$ in $\partial\mathbf{H}^4$ to the band system $b_i\,(i=1,2,\dots,n,\dots)$ in $\mathbf{H}^4$. Let

$$d_*=\cup_{i=1}^{+\infty} d_i,\quad \bar{L}^D=d_*\cup_{j=1}^{+\infty} h_j.$$

The *closed exteriors* of d_* and $\bar{L}^D$ in $\mathbf{H}^4$ are the 4D manifolds

$$E(d_*)=\mathrm{cl}(\mathbf{H}^4\setminus N(d_*)),\quad E(\bar{L}^D)=\mathrm{cl}(\mathbf{H}^4\setminus N(\bar{L}^D))$$

for regular neighborhoods $N(d_*), N(\bar{L}^D)$ of $d_*, \bar{L}^D$ in $\mathbf{H}^4$, respectively. Then the closed exterior $E(\bar{L}^D)$ is diffeomorphic to the closed exterior $E(d_*)$ which is considered as a 4D manifold obtained from $\mathbf{H}^4$ by attaching a discrete family of disjoint 1-handles along $\partial\mathbf{H}^3$. The closed exterior $E(L^D)$ is obtained from $E(\bar{L}^D)$ by adding a discrete system of disjoint 2-handles arising from the the band system $b_j^0\,(j=1,2,\dots,n,\dots)$. This completes the proof of Lemma 5.1. *square*

In the following lemma, (1) is essentially a consequence of Lemma 4.1, and (2) is a generalization of [9, Lemma 3.3].

Lemma 5.2. Assume that the 2-complex of a group presentation

$$< x_1, x_2, \dots x_n, \dots \,|\, r_1, r_2, \dots, r_m, \dots >$$

is a locally finite contractible 2-complex. Then there is a ribbon disk-link L^D with components $K_i^D\,(i=1,2,\dots,n,\dots)$ in H^4 such that

(1) the fundamental group $\pi_1(\mathbf{H}^4 \setminus L^D, v)$ is isomorphic to the free group with basis $x_i\,(i=1,2,\dots,n,\dots)$ by an isomorphism sending a meridian system of $K_i^D\,(i=1,2,\dots,n,\dots)$ to the relator system $r_i\,(i=1,2,\dots,n,\dots)$, and

(2) For every sublink L_1^D of L^D, the second homotopy group $\pi_2(\mathbf{H}^4 \setminus L_1^D, v)=0$.

Proof of Lemma 5.2. For (1), by Lemma 4.1 there is a ribbon S^2-link L with components $K_i\,(i=1,2,\dots,n,\dots)$ in $\mathbf{R}^4$ such that the fundamental group $\pi_1(\mathbf{R}^4 \setminus L, v)$ is isomorphic to the free group with basis $x_i\,(i=1,2,\dots,n,\dots)$ by an isomorphism sending a meridian system of $K_i\,(i=1,2,\dots,n,\dots)$ to the relator system $r_i\,(i=1,2,\dots,n,\dots)$. Then the ribbon S^2-link L in $\mathbf{R}^4$ is the union of a ribbon disk-link L^D with components $K_i^D\,(i=1,2,\dots,n,\dots)$ in $\mathbf{H}^4$ and the image $\alpha(L^D)$ with components $\alpha(K_i^D)\,(i=1,2,\dots,n,\dots)$ in $\alpha(\mathbf{H}^4)$ by the reflection α (see [1] for homotopical deformations of 1-handles and [10, II:Lemma5.11] for an α-invariant deformation of L). By [9, Lemma 3.3], the inclusion $(\mathbf{H}^4, L^D) \to (R^4, L)$ induces an isomorphism

$$\pi_1(\mathbf{H}^4 \setminus L^D, v) \to \pi_1(\mathbf{R}^4 \setminus L, v).$$

Thus, the ribbon disk-link L^D has the property (1).

The proof of (2) is analogous to the proof of [9, Lemma 3.3], but for completeness the proof for the infinite ribbon disk-link L^D in $\mathbf{H}^4$ is given. Let L_1^D be a sublink of L^D. Let $\tilde{S}$ be an immersed 2-sphere in the closed exterior $E(L_1^D)$, which is considered as an immersed 2-sphere in the closed exterior $E(L^D)$ by taking the ribbon disk system $L^D \setminus L_1^D$ in a thin boundary collar of $\mathbf{H}^4$. Let L be a ribbon S^2-link in $\mathbf{R}^4$ obtained from L^D as the double by α. Let Y be the 4D manifold obtained from $\mathbf{R}^4$

by surgery along L. By Lemma 4.1, the second homotopy group $\pi_2(Yv) = 0$. Hence the immersed sphere $\tilde{S}$ in $E(L^D)$ bounds an immersed 3-ball $\tilde{B}$ in Y. Let $k(L)$ be the loop system in Y occurring from the surgery along L. By general position, the loop system $k(L)$ meets transversely the immersed 3-ball $\tilde{B}$ in a finite set, say an s point set. Then there is a compact s-punctured immersed 3-ball $\tilde{B}^{(s)}$ in the closed exterior $E(L)$ of L in S^4 such that $\partial\tilde{B}^{(s)} \supset \tilde{S}$ and $\partial\tilde{B}^{(s)} \setminus \tilde{S}$ is a 2-sphere system $S_i\,(i = 1, 2, \ldots, s)$ in the boundary $\partial E(L)$. Note that the closed exterior $E(L)$ is the union of the closed exterior $E(L^D)$ and the other copy $\alpha(E(L^D))$, the inmage of $E(L^D)$ by the reflection α. By transforming the intersection part $\tilde{B}^{(s)} \cap \alpha(E(L^D))$ into $E(L^D)$ by the reflection α, the punctured immersed 3-ball $\tilde{B}^{(s)}$ is taken in the compact exterior $E(L^D)$ so that the (possibly singular) 2-spheres $S_i\,(i = 1, 2, \ldots, s)$ are in $L^D \times S^1 \subset \partial E(L)$. Since each component of $L^D \times S^1$ is aspherical, the immersed 2-speres $S_i\,(i = 1, 2, \ldots, s)$ bounds singular 3-balls in $L^D \times S^1$. This means that the immersed sphere $\tilde{S}$ is null-homotopic in $E(L^D) \subset E(K_1^D)$. Hence $\pi_2(E(L_1^D), v) = 0$. This completes the proof of Lemma 5.2. □

The proof of Theorem 2.1 is done as follows.

5.3: Proof of Theorem 2.1. Let P be a contractible locally finite 2-complex which is the 2-complex of the group presentation $< x_1, x_2, \ldots, x_n, \ldots \,|\, r_1, r_2, \ldots, r_m, \cdots >$. Let P_1 be any finite connected subcomplex of P, which is the 2-complex of the group presentation given by a finite sub presentation $< x_{i_1}, x_{i_2}, \ldots, x_{i_s} \,|\, r_{j_1}, r_{j_2}, \ldots, r_{j_t} >$. Let P_2 be the 2-complex of the group presentation given by a finite sub presentation $< x_1, x_2, \ldots x_n, \ldots \,|\, r_{j_1}, r_{j_2}, \ldots, r_{j_t} >$. Then the 2-complex P_2 is homotopy equivalent to the 2-complex obtained from P_1 by joining a half straight line with circles attached which correspond to the generators x_i for all i except for $i_1, i_2, \cdots_s$. Thus, P_1 is aspherical if and only if P_2 is aspherical. By Lemma 5.2 (1) , let L^D be a ribbon disk-link with components $K_j^D\,(j = 1, 2, \ldots, n, \ldots)$ in $\mathbf{H}^4$ such that the fundamental group $\pi_1(\mathbf{H}^4 \setminus L^D, v)$ is isomorphic to the free group with basis $x_i\,(i = 1, 2, \ldots, n, \ldots)$ by an isomorphism sending a meridian system of $K_j^D\,(j = 1, 2, \ldots, n, \ldots)$ to the relator system $r_j\,(j = 1, 2, \ldots, n, \ldots)$. Let L_2^D be a ribbon disk-link in $\mathbf{H}^4$ with the components K_j^D for all j except for $j_1, j_2, \ldots, j_t$. By van Kampen theorem, the fundamental group $\pi_1(\mathbf{H}^4 \setminus L_2^D, v)$ has the group presentation $< x_1, x_2, \ldots x_n, \ldots \,|\, r_{j_1}, r_{j_2}, \ldots, r_{j_t} >$. By Lemmas 5.1 and 5.2 (2), the closed exterior $E(L_2^D)$ is aspherical. On the other hand, by Lemma 5.2 (2) the closed exterior $E(L^D)$ is homotopy equivalent to a 1-complex R which is a straight line with loops attached corresponding to the generators x_i for all i and the closed exterior $E(L_2^D)$ is homotopy equivalent to the 2-complex obtained from R by attaching the meridiak 2-cells of the components of L^D corresponding to the relators $r_{j_1}, r_{j_2}, \ldots, r_{j_t}$. Hence the closed exterior $E(L_2^D)$ is homotopy equivalent to the 2-complex P_2. Thus, P_2 is aspherical and hence P_1 is aspherical. This completes the proof of Theorem 2.1. □

The asphericity of the closed exterior of every finite ribbon disk-link in $\mathbf{H}^4$ is shown in [9] by using the results of the smooth unknotting conjecture for a surface-link in [3, 4, 5] and the 4D smooth Poincaré conjecture in [6, 7]. Actually, this result holds without use of them since all the results of this paper containing the following corollary are done without use of them.

Corollary 5.4. The closed exterior $E(L^D)$ of every (finite or infinite) ribbon disk-link L^D in $\mathbf{H}^4$ is aspherical.

Proof of Corollary 5.4. By Lemma 5.1, the closed exterior $E(L^D)$ is homotopy equivalent to a connected 2-complex P and made contractible by attaching meridian disks of L^D, so that the 2-complex P is a connected subcomplex of a contractible 2-complex and aspherical by Theorem 1.1. This completes the proof of Corollary 5.4. □

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