Kervaire conjecture on weight of group via fundamental group of ribbon sphere-link

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ABSTRACT

Kervaire conjecture that the weight of the free product of every non-trivial group and the infinite cyclic group is not one is confirmed by confirming Conjecture Z of a knot exterior.

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1. Introduction

A *weight system* of a group G is a system of elements w_i, $(i = 1, 2, \ldots, n)$ of G such that the normal closure of the elements w_i, $(i = 1, 2, \ldots, n)$ (=: the smallest normal subgroup generated by w_i, $(i = 1, 2, \ldots, n)$) is equal to G. The *weight* of a group G is the least cardinal number of a weight system of G. *Kervaire's conjecture* on the weight of a group is the following conjecture (see Kervaire [6], Magnus-Karrass-Solitar [8, p. 403]):

Kervaire Conjecture. The free product $G * \mathbf{Z}$ of every non-trivial group G and the infinite cyclic group $\mathbf{Z}$ is not of weight one.

Some partial confirmations of this conjecture are known. In this paper, the result of Klyachko [7] that $G * \mathbf{Z}$ is not of weight one for every non-trivial torsion-free group

G is used. González-Acuña and Ramírez in [2] showed that it is sufficient to confirm this conjecture for every non-trivial finitely presented group, and further showed that this conjecture is equivalent to it Conjecture Z of a knot exterior. To explain this conjecture, some terminologies are fixed. A *knot exterior* is a compact 3-manifold $E = \mathrm{cl}(S^3 \setminus N(k))$ for a tubular neighborhood $N(k)$ of a polygonal knot k in the 3-sphere S^3. Let Fbe a compact connected orientable non-separating proper surface of E. Let $E(F) = \mathrm{cl}(E \setminus F \times I)$ be the compact piecewise-linear 3-manifold for a normal line bundle $F \times I$ of F in $E(F)$ where $I = [-1, 1]$. Let $\hat{E}(F)$ be the 3-complex obtained from $E(F)$ by adding the cone $\mathrm{Cone}(v, F \times \partial I)$ over the base $F \times \partial I$ with vertex v. The 3-complex $\hat{E}(F)$ is also obtained from E by shrinking the normal line bundle $F \times I$ into the vertex v.

Conjecture Z. The fundamental group $\pi_1(\hat{E}(F), v)$ is always isomorphic to $\mathbf{Z}$.

See [1, 2, 9] for some investigations of this surface F and some partial confirmations. In this paper, Kervaire conjecture is confirmed by confirming this Conjecture Z.

Theorem. Conjecture Z is true. .

The proof of Theorem is done by showing that the fundamental group $\pi_1(\hat{E}(F), v)$ is always torsion-free, which comes from torsion-freeness of the fundamental group of a ribbon S^2-link claimed in [4]. Conceptially, it can be understood that this result comes from Whitehead aspherical conjecture saying that every connected subcomplex of every aspherical 2-complex is aspherical (see [4, 5]). Then $\pi_1(\hat{E}(F), v) \cong \mathbf{Z}$ is concluded by the result of [7].

In the first draft of this research, the author tried to show that every finitely presented group G with free product group $G * \mathbf{Z}$ of weight one is torsion-free. This trial succeeds for a group G of deficiency 0, but failed for a group G of negative deficiency. The main point of this failure is the attempt to construct a finitely presented group of deficiency 0 from the group of negative deficiency, which forced the author to show that G is torsion-free while the deficiency remains negative. Fortunately, in Conjecture Z, the 3-complex $\hat{E}(F)$ which is the object to be considered is simple, so it could be done.

The proof of Theorem is done in Section 2.

2. Proof of Theorem

The proof of Theorem is done as follows, but the main point of the proof is how a finite contractible 2-complex is constructed to contain, as a subcomplex, a 2-complex which is homotopy equivalent to the 3-complex $\hat{E}'$.

Proof of Theorem. Let $\hat{E}(F)'$ be the 3-complex obtained from $E(F)$ by shrinking $F\times 1$ and $F\times(-1)$ into distinct vertexes v_+ and v_-, respectively. The 3-complex $\hat{E}(F)$ is homotopy equivalent to a bouquet $X = \hat{E}(F)' \vee S^1$ of $\hat{E}(F)'$ and the circle S^1. Let Y be the 3-complex obtained from X by attaching of a 2-cell arising from attaching a meridian disk to E. Then $\pi_1(Y,v)$ is obtained from $\pi_1(X,v) = \pi_1(\hat{E},v) * \mathbf{Z}$ by adding a relation and $\pi_1(Y,v) = \{1\}$. If $\pi_1(\hat{E}(F),v)$ is torsion-free, then $\pi_1(\hat{E}(F)',v)$ is torsion-free. By the result of [7], $\pi_1(\hat{E}(F)',v) = \{1\}$ and $\pi_1(\hat{E}(F),v) \cong \mathbf{Z}$. To complete the proof, it sufficies to show that $\pi_1(\hat{E}(F),v)$ is torsion-free, Collapse F into a triangulated graph γ by using that F is a bounded surface. Enlarge the fiber I of a normal line bundle $F \times I$ of F in E into a fiber $\bar{I}$ of a normal line bundle $F \times \bar{I}$ of F in E so that $I \subset \bar{I} \setminus \partial\bar{I}$. Let $\bar{I}^c = \mathrm{cl}(\bar{I} \setminus I)$. Let $E(F)^- = \mathrm{cl}(E \setminus F \times \bar{I})$. Collapse $F \times \bar{I}^c$ into $\gamma \times \bar{I}^c$. Triangulate $\gamma \times \bar{I}^c$ without introducing new vertexes. The 3-complex $\hat{E}(F)$ is collapsed into a finite 3-complex

$$E(F)^- \cup \gamma \times \bar{I}^c \cup \mathrm{Cone}(v, \gamma \times \partial I)$$

and thus collapsed into a finite 2-complex

$$P = P^- \cup \gamma \times \bar{I}^c \cup \mathrm{Cone}(v, \gamma \times \partial I)$$

for any 2-complex P^- collapsed from $E(F)^-$. This 2-complex P is a subcomplex of a 3-complex

$$Q = \mathrm{Cone}(v, P^- \cup \gamma \times \bar{I}^c).$$

Since every 2-complex of $\gamma \times \bar{I}^c$ contains at most one 1-simplex of $\gamma \times \partial I$, every 3-simplex of $\mathrm{Cone}(v, \gamma\times\bar{I}^c)$ contains at most one 2-simplex of $\mathrm{Cone}(v, \gamma\times\partial I)$. Collapse every 3-simplex of $\mathrm{Cone}(v, \gamma \times \bar{I}^c)$ from a 2-face containing v and not belonging to $\mathrm{Cone}(v, \gamma \times \partial I)$. Then collapse every 3-simplex of $\mathrm{Cone}(v, P^-)$ from any 2-face containing v. Thus, the 3-complex Q is collapsed into a finite 2-complex C containing the 2-complex P as a subcomplex. Since Q is contractible, C is a finite contractible 2-complex. By the argument of [4], the group $\pi_1(P,v)$ is a torsion-free group, actually a ribbon S^2-knot group for $H_1(P;\mathbf{Z}) \cong \mathbf{Z}$. Thus, the group $\pi_1(\hat{E}(F),v)$ is torsion-free since $\pi_1(\hat{E}(F),v)$ is isomorphic to $\pi_1(P,v)$. This completes the proof of Theorem. □

The result that the fundamental group of a ribbon S^2-link is torsion-free can be shown without use of the smooth 4D Poincaré conjecture and the smooth unknotting conjecture for an S^2-link as it is observed in [5], although the proof in [4] uses them. On the other hand, the result of [3, Lemma 3.4] is still basic to this proof.

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