Isotropicity of surfaces in Lorentzian 4-manifolds with zero mean curvature vector

Naoya Ando (Kumamoto University)

Contents

1. Mixed-type structures of 4-dimensional Lorentzian vector spaces

X: an oriented 4-dimensional vector space,

h_X: a symmetric and indefinite bilinear form of X with signature (3,1),

(e_1, e_2, e_3, e_4): an ordered basis of X giving the orientation of X s.t.

$$h_X(e_i, e_j) = 0 \quad (i \neq j),$$
$$h_X(e_i, e_i) = 1 \quad (i = 1, 2, 3), \qquad h_X(e_4, e_4) = -1.$$

$\mathcal{B}_X$: the set of ordered bases of X as (e_1, e_2, e_3, e_4).

$\theta_{ij} := e_i \wedge e_j$,

$\hat{h}_X$: a bilinear form of $\bigwedge^2 X$ defined by

$$\hat{h}_X(\theta_{ij}, \theta_{kl}) = h_X(e_i, e_k) h_X(e_j, e_l) - h_X(e_i, e_l) h_X(e_j, e_k).$$

$\Theta_{\pm,1} := \frac{1}{\sqrt{2}}(\theta_{12} \pm \theta_{34})$, $\Theta_{\pm,2} := \frac{1}{\sqrt{2}}(\theta_{13} \pm \theta_{42})$, $\Theta_{\pm,3} := \frac{1}{\sqrt{2}}(\theta_{14} \pm \theta_{23})$.

These are light-like and we have

$$\hat{h}_X(\Theta_{\varepsilon,i}, \Theta_{\varepsilon',j}) = 0 \quad (\varepsilon, \varepsilon' \in \{+,-\},\ 1 \leqq i < j \leqq 3),$$
$$\hat{h}_X(\Theta_{+,i}, \Theta_{-,i}) = 1 \quad (i = 1, 2), \qquad \hat{h}_X(\Theta_{+,3}, \Theta_{-,3}) = -1.$$

We see that $\hat{h}_X$ is a symmetric and indefinite bilinear form of $\bigwedge^2 X$ with signature (3,3).

Remark

- If h_X is positive-definite,
  then noticing a double covering $SO(4) \longrightarrow SO(3) \times SO(3)$,
  we have a decomposition $\bigwedge^2 X = \bigwedge^2_+ X \oplus \bigwedge^2_- X$,
  where $\bigwedge^2_+ X$, $\bigwedge^2_- X$ are subspaces of $\bigwedge^2 X$ with $\dim \bigwedge^2_\pm X = 3$ s.t.

$$\bigwedge\nolimits^2_+ X = \langle E_{+,1}, E_{+,2}, E_{+,3} \rangle, \quad \bigwedge\nolimits^2_- X = \langle E_{-,1}, E_{-,2}, E_{-,3} \rangle.$$

- If h_X has signature (2,2),
  then noticing a double covering $SO_0(2,2) \longrightarrow SO_0(1,2) \times SO_0(1,2)$,
  we have $\bigwedge^2 X = \bigwedge^2_+ X \oplus \bigwedge^2_- X$,
  where $\bigwedge^2_+ X$, $\bigwedge^2_- X$ are subspaces of $\bigwedge^2 X$ with $\dim \bigwedge^2_\pm X = 3$ s.t.

$$\bigwedge\nolimits^2_+ X = \langle E_{-,1}, E_{+,2}, E_{+,3} \rangle, \quad \bigwedge\nolimits^2_- X = \langle E_{+,1}, E_{-,2}, E_{-,3} \rangle.$$

K: a linear transformation of X.

We call K a *mixed-type structure* of X

if K has invariant subspaces $X_\pm$ of X with $\dim X_\pm = 2$ s.t.

- $X_\pm$ are eigenspaces of K^2 so that ∓ 1 are the corresponding eigenvalues, respectively,
- $K|_{X_-}$ is not the identity map.

K: a mixed-type structure of X.

We say that K is *compatible with* h_X if K satisfies

- each nonzero element of X_+ is space-like,
- $(K|_{X_\pm})^* h_X = \pm h_X$,
- $X_\pm$ are perpendicular to each other.

K: a mixed-type structure of X compatible with h_X.
We say that K is *compatible with the orientation of* X if $(e_1, K(e_1), e_3, K(e_3)) \in \mathcal{B}_X$ for any unit vector $e_1 \in X_+$ and any space-like and unit vector $e_3 \in X_-$.

K_+: a mixed-type structure of X compatible with h_X and the orientation of X.
We see that

$$\frac{1}{\sqrt{2}}(e_1 \wedge K_+(e_1) + e_3 \wedge K_+(e_3)) \qquad (\sharp 1)$$

is light-like and determined by K_+, and does not depend on the choice of a pair (e_1, e_3).

K_-: a mixed-type structure of X s.t.

- K_- is compatible with h_X,
- K_- is not compatible with the orientation of X.

We see that

$$\frac{1}{\sqrt{2}}(e_1 \wedge K_-(e_1) + e_3 \wedge K_-(e_3)) \qquad (\sharp 2)$$

is light-like and determined by K_-, and does not depend on the choice of (e_1, e_3).

Example

$(e_1, e_2, e_3, e_4) \in \mathcal{B}_X$,

K_+: a linear transformation of X defined by

$$(K_+(e_1)\ K_+(e_2)\ K_+(e_3)\ K_+(e_4)) = (e_1\ e_2\ e_3\ e_4) \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$\Longrightarrow\ (K_+^2(e_1)\ K_+^2(e_2)\ K_+^2(e_3)\ K_+^2(e_4)) = (e_1\ e_2\ e_3\ e_4) \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$X_+ := \langle e_1, e_2\rangle,\ \ X_- := \langle e_3, e_4\rangle.$

$\Longrightarrow$ K_+ is a mixed-type structure, that is, K_+ satisfies the above conditions:

- $K_+(X_+) = X_+$, $K_+(X_-) = X_-$,
- $X_\pm$ are the ∓ 1-eigenspaces of K_+^2,
- $K_+|_{X_-} \neq \mathrm{id}_{X_-}$.

In addition, K_+ is compatible with h_X:

- $(K_+|_{X_+})^* h_X = h_X$ $\ (h_X(K_+(e_i), K_+(e_j)) = h_X(e_i, e_j)$ for $i, j = 1, 2)$,
- $(K_+|_{X_-})^* h_X = -h_X$ by

$$\begin{aligned} h_X(K_+(e_3), K_+(e_3)) &= h_X(e_4, e_4) = -h_X(e_3, e_3), \\ h_X(K_+(e_4), K_+(e_4)) &= h_X(e_3, e_3) = -h_X(e_4, e_4), \\ h_X(K_+(e_3), K_+(e_4)) &= h_X(e_4, e_3) = -h_X(e_3, e_4), \end{aligned}$$

- $X_+ \perp X_-$.

Since $(e_1, K_+(e_1), e_3, K_+(e_3)) = (e_1, e_2, e_3, e_4) \in \mathcal{B}_X$, K_+ is compatible with the orientation of X.

K_-: a linear transformation of X defined by

$$(K_-(e_1)\ K_-(e_2)\ K_-(e_3)\ K_-(e_4)) = (e_1\ e_2\ e_3\ e_4)\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

$\Longrightarrow$ K_- is a mixed-type structure of X s.t.

- K_- is compatible with h_X,
- K_- is not compatible with the orientation of X.

X': an oriented 2-dimensional subspace of X s.t. each nonzero element of X' is space-like.

$\Longrightarrow$ $\exists K_+$: a mixed-type structure of X compatible with h_X and the orientation of X s.t.

- $X' = X_+$,
- for a nonzero vector $e_1 \in X_+$, $(e_1, K_+(e_1))$ gives the orientation of $X' = X_+$.

$\exists K_-$: a mixed-type structure of X compatible with h_X and not compatible with the orientation of X s.t.

- $X' = X_+$,
- for a nonzero vector $e_1 \in X_+$, $(e_1, K_-(e_1))$ gives the orientation of $X' = X_+$.

X'': an oriented 2-dimensional subspace of X which has a time-like vector of X.

$\Longrightarrow$ $\exists K_+$: a mixed-type structure of X compatible with h_X and the orientation of X s.t.

- $X'' = X_-$,
- for a space-like vector $e_3 \in X_-$, $(e_3, K_+(e_3))$ gives the orientation of $X'' = X_-$.

$\exists K_-$: a mixed-type structure of X compatible with h_X and not compatible with the orientation of X s.t.

- $X'' = X_-$,
- for a space-like vector $e_3 \in X_-$, $(e_3, -K_-(e_3))$ gives the orientation of $X'' = X_-$.

2. Space-like surfaces in Lorentzian 4-manifolds

M: a manifold,

E: an oriented vector bundle over M of rank 4.

h: an indefinite metric of E with signature $(3, 1)$,

∇: a connection of E s.t. $\nabla h = 0$.

$\hat{h}$: the metric of $\bigwedge^2 E$ induced by h.

$\Longrightarrow$ $\hat{h}$ has signature $(3, 3)$.

$\hat{\nabla}$: the connection of $\bigwedge^2 E$ induced by ∇.

$\Longrightarrow$ $\hat{\nabla}\hat{h} = 0$.

E': an oriented subbundle of E of rank 2 s.t. each nonzero element of each fiber of E is space-like.

Then E' defines *mixed-type structures* $K_\pm$ of E, i.e., sections of $\mathrm{End}\,(E)$ which give mixed-type structures of the fiber E_a of E defined by the fiber E'_a of E' for each $a \in M$.

Mixed-type structures $K_\pm$ of E defined by E' give light-like sections $\Theta_\pm$ of $\bigwedge^2 E$ by ($\sharp$1) and ($\sharp$2).

M: a Riemann surface,

N: an oriented 4-dimensional Lorentzian manifold,

$F : M \longrightarrow N$: a space-like and conformal immersion.

$E := F^*TN$.

We see that F gives a subbundle E' of E by $E' = F^*(dF(TM))$.

$K_{F,\pm}$: the mixed-type structures of E given by E'.

We call each of light-like sections $\Theta_{F,\pm}$ of $\bigwedge^2 E$ given by $K_{F,\pm}$ a *lift* of F.

$w = u + \sqrt{-1}v$: a local complex coordinate of M,

$$T_1 := dF\left(\frac{\partial}{\partial u}\right), \ \ T_2 := dF\left(\frac{\partial}{\partial v}\right).$$

Suppose that F has zero mean curvature vector.

$\Longrightarrow \ \nabla_{T_1}T_1 + \nabla_{T_2}T_2 = 0.$

- We say that F is *isotropic*
  if we can choose w s.t.

$$h(\sigma(T_1, T_1), \sigma(T_1, T_1)) = -h(\sigma(T_1, T_2), \sigma(T_1, T_2)),$$
$$h(\sigma(T_1, T_1), \sigma(T_1, T_2)) = 0.$$

- We say that F is *strictly isotropic*
  if we can choose w s.t. $K_{F,+}\sigma(T_1, T_1) = \sigma(T_1, T_2)$.

Remark If F is strictly isotropic, then F is isotropic.

$\Psi := dF(\partial/\partial w)$.

$\Longrightarrow \ \overline{\nabla}_{\partial/\partial w}(\Psi dw) = \sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right) dw.$

We can define a complex quartic differential Q on M by

$$Q := h\left(\sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right),\ \sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right)\right) dw \otimes dw \otimes dw \otimes dw.$$

If N is a 4-dimensional Lorentzian space form,

then we see by the equations of Codazzi that Q is holomorphic.

Theorem

N: *an oriented* 4*-dimensional Lorentzian manifold,*

M: *a Riemann surface,*

$F : M \longrightarrow N$: *a space-like and conformal immersion with zero mean curvature vector.*

Then F *is isotropic if and only if one the following holds:*

(a) $Q \equiv 0$;

(b) F *is strictly isotropic, by rechoosing the orientation of* N *if necessary.*

Proof

ν_1, ν_2: normal vector fields of F s.t.

$$h(\nu_1,\nu_1)=1, \quad h(\nu_2,\nu_2)=-1, \quad h(\nu_1,\nu_2)=0.$$

We represent $\sigma(T_k,T_l)$ as $\sigma(T_k,T_l)=c^1_{kl}\nu_1+c^2_{kl}\nu_2$.

Suppose that F is isotropic.

Then we obtain $((c^1_{11})^2-(c^2_{11})^2)((c^1_{12})^2-(c^2_{11})^2)=0$.

- If $(c^1_{11})^2=(c^2_{11})^2$, then we have $(c^2_{11},c^2_{12})=\pm(c^1_{11},c^1_{12})$, i.e., $Q\equiv 0$.
- If $(c^1_{12})^2=(c^2_{11})^2$, then we have $(c^2_{11},c^2_{12})=\pm(c^1_{12},c^1_{11})$ and then F is strictly isotropic, by rechoosing the orientation of N if necessary.

If either $Q\equiv 0$ or F is strictly isotropic, then F is isotropic. □

Theorem N, M, F: *as in the previous theorem.*
Then the following are mutually equivalent:

(a) $Q \equiv 0$;

(b) $\hat{h}(\hat{\nabla}_{T_k}\Theta_{F,+}, \hat{\nabla}_{T_k}\Theta_{F,+}) = 0$,
$\hat{h}(\hat{\nabla}_{T_k}\Theta_{F,-}, \hat{\nabla}_{T_k}\Theta_{F,-}) = 0$,
$\hat{h}(\hat{\nabla}_{T_k}\Theta_{F,+}, \hat{\nabla}_{T_k}\Theta_{F,-}) = 0$ *for* $k = 1, 2$;

(c) *the second fundamental form is light-like or zero.*

Remark

Suppose that F is strictly isotropic.
Then we obtain $\hat{h}(\hat{\nabla}_{T_k}\Theta_{F,+}, \hat{\nabla}_{T_k}\Theta_{F,-}) = 0$,
while we do not necessarily obtain $\hat{h}(\hat{\nabla}_{T_k}\Theta_{F,\varepsilon}, \hat{\nabla}_{T_k}\Theta_{F,\varepsilon}) = 0$ for $\varepsilon \in \{+, -\}$.

Remark

N: an oriented 4-dimensional Riemannian or neutral manifold,

$F : M \longrightarrow N$: a space-like and conformal immersion with zero mean curvature vector.

Then by the bundle decomposition $\bigwedge^2 E = \bigwedge^2_+ E \oplus \bigwedge^2_- E$ with $E = F^*TN$, we have $\hat{h}(\hat{\nabla}_{T_k}\Theta_{F,+}, \hat{\nabla}_{T_k}\Theta_{F,-}) = 0$.

We see that F is strictly isotropic if and only if a suitable one of $\Theta_{F,\pm}$ is horizontal.

Remark

N: an oriented 4-dimensional neutral manifold,

M: a Lorentz surface,

$F : M \longrightarrow N$: a time-like and conformal immersion with zero mean curvature vector.

Then we have analogues of results mentioned in the previous remark.

In addition, if F is isotropic and if none of the covariant derivatives of $\Theta_{F,\pm}$ become zero,

then the covariant derivatives are light-like and the second fundamental form of F is light-like or zero.

3. Time-like surfaces in Lorentzian 4-manifolds

M: a Lorentz surface,

N: an oriented 4-dimensional Lorentzian manifold,

$F : M \longrightarrow N$: a time-like and conformal immersion,

$E := F^*TN$.

We see that F gives a subbundle E'' of E by $E'' = F^*(dF(TM))$.

E': the subbundle of E given by the orthogonal complement of E'',

$K_{F,\pm}$: the mixed-type structure of E given by E'.

We call each of light-like sections $\Theta_{F,\pm}$ of $\bigwedge^2 E$ given by $K_{F,\pm}$ a *lift* of F.

$w = u + jv$: a local paracomplex coordinate of M,

$T_1 := dF\left(\frac{\partial}{\partial u}\right)$, $T_2 := dF\left(\frac{\partial}{\partial v}\right)$.

Suppose that F has zero mean curvature vector.

$\Longrightarrow \nabla_{T_1} T_1 = \nabla_{T_2} T_2$.

- We say that F is *isotropic*
  if we can choose w s.t.

$$h(\sigma(T_1, T_1), \sigma(T_1, T_1)) = h(\sigma(T_1, T_2), \sigma(T_1, T_2)),$$
$$h(\sigma(T_1, T_1), \sigma(T_1, T_2)) = 0.$$

- We say that F is *strictly isotropic*
  if we can choose w s.t. $K_{F,+}\sigma(T_1, T_1) = \sigma(T_1, T_2)$.

Remark If F is strictly isotropic, then F is isotropic.

$$\Psi := dF\left(\frac{\partial}{\partial w}\right) = \frac{1}{2}(T_1 + jT_2).$$

$$\implies \overline{\nabla}_{\partial/\partial w}(\Psi dw) = \sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right) dw.$$

We can define a complex quartic differential Q on M by

$$Q := h\left(\sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right),\ \sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right)\right) dw \otimes dw \otimes dw \otimes dw.$$

We see that $Q \equiv 0$ if and only if F is totally geodesic.

If N is a 4-dimensional Lorentzian space form,

then we see by the equations of Codazzi that Q is holomorphic.

Theorem

N: an oriented 4-dimensional Lorentzian manifold,

M: a Lorentz surface,

$F : M \longrightarrow N$: a time-like and conformal immersion with zero mean curvature vector.

Then F is isotropic if and only if F is strictly isotropic, by rechoosing the orientation of N if necessary.

Remark

Suppose that F is strictly isotropic.

Then we obtain $\hat{h}(\hat{\nabla}_{T_k}\Theta_{F,+}, \hat{\nabla}_{T_k}\Theta_{F,-}) = 0$,

while we do not necessarily obtain $\hat{h}(\hat{\nabla}_{T_k}\Theta_{F,\varepsilon}, \hat{\nabla}_{T_k}\Theta_{F,\varepsilon}) = 0$ for $\varepsilon \in \{+, -\}$.

If $\hat{h}(\hat{\nabla}_{T_k}\Theta_{F,\varepsilon}, \hat{\nabla}_{T_k}\Theta_{F,\varepsilon'}) = 0$ for $k = 1, 2$ and $\varepsilon, \varepsilon' \in \{+, -\}$,

then we have $\sigma(T_1, T_2) = \pm\sigma(T_1, T_1)$.

4. The images of the lifts by the curvature tensor

R: the curvature tensor of ∇:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$\hat{R}$: the curvature tensor of $\hat{\nabla}$.

$$\Longrightarrow \ \hat{R}(X_1, X_2)(Y_1 \wedge Y_2) = (R(X_1, X_2)Y_1) \wedge Y_2 + Y_1 \wedge R(X_1, X_2)Y_2.$$

M: a Riemann surface,

$F : M \longrightarrow N$: a space-like and conformal immersion,

(e_1, e_2): a local ordered orthonormal frame field of TM giving the orientation of M.

Theorem (A, 2020)

$F : M \longrightarrow N$: *a space-like and conformal immersion with zero mean curvature vector s.t.* $\hat{R}(e_1, e_2)\Theta_{F,\pm} = 0$.

Then the following hold:

(a) Q *is holomorphic;*

(b) *if* $Q \equiv 0$ *and if* $d\omega^{\perp} = 0$ *with* $\omega^{\perp} := h(\nabla e_3, e_4)$,
then F *is strictly isotropic, by rechoosing the orientation of* N *if necessary;*

(c) *if* F *is strictly isotropic and if* F *is not totally geodesic on any open set of* M, *then the connection forms* $\omega := h(\nabla e_1, e_2)$ *and* $\omega^{\perp}$ *satisfy* $d * \omega = 0$ *and* $d\omega^{\perp} = 0$ *for a suitable* (e_1, e_2), *and the 2nd fundamental form of* F *is constructed by a solution of an over-determined system s.t. the compatibility condition is given by* $d * \omega = 0$ *and* $d\omega^{\perp} = 0$.

Proof of (b) of the theorem

Suppose $Q \equiv 0$.

$\Longrightarrow$ The shape operator of a light-like normal vector field ν of F vanishes.

U: a neighborhood of a point of M where the 2nd fundamental form does not vanish,

$(\tilde{u}, \tilde{v})$: local coordinates on U s.t. $\partial/\partial\tilde{u}$, $\partial/\partial\tilde{v}$ are in principal directions of F w.r.t. a light-like normal vector field ι satisfying $h(\iota, \nu) = -1$.

The induced metric g on M by F is represented as $g = \tilde{A}^2 d\tilde{u}^2 + \tilde{B}^2 d\tilde{v}^2$.

Since $\hat{R}(e_1, e_2)\Theta_{F,\pm} = 0$, we have $(R(e_1, e_2)e_i)^\perp = 0$ $(i = 1, 2)$.

Since $d\omega^\perp = 0$, we can find a function γ defined on a neighborhood of each point of M s.t. $\omega^\perp = -d\gamma$.

In addition, we obtain $\gamma_{\tilde{u}} = h(\nabla_{\partial/\partial\tilde{u}}\iota, \nu)$ and $\gamma_{\tilde{v}} = h(\nabla_{\partial/\partial\tilde{v}}\iota, \nu)$.

k: a positive-valued function on U s.t. k and $-k$ are principal curvatures of F w.r.t. ι.

Then using $(R(e_1, e_2)e_i)^{\perp} = 0$, $\gamma_{\tilde{u}} = h(\nabla_{\partial/\partial\tilde{u}}\iota, \nu)$ and $\gamma_{\tilde{v}} = h(\nabla_{\partial/\partial\tilde{v}}\iota, \nu)$, we obtain $(ke^{\gamma}\tilde{A}^2)_{\tilde{v}} = 0$ and $(ke^{\gamma}\tilde{B}^2)_{\tilde{u}} = 0$, which mean $ke^{\gamma}\tilde{A}^2 = \phi^2$ and $ke^{\gamma}\tilde{B}^2 = \psi^2$ for positive-valued functions $\phi = \phi(\tilde{u})$, $\psi = \psi(\tilde{u})$.

u, v: functions of one variable $\tilde{u}$, $\tilde{v}$ respectively s.t. $\dfrac{du}{d\tilde{u}} = \phi$, $\dfrac{dv}{d\tilde{v}} = \psi$.

$\Longrightarrow$ (u, v) are isothermal coordinates of M w.r.t. g and $w = u + \sqrt{-1}v$ is a local complex coordinate of M.

A_ι: the shape operator of F w.r.t. ι.

Then we can suppose

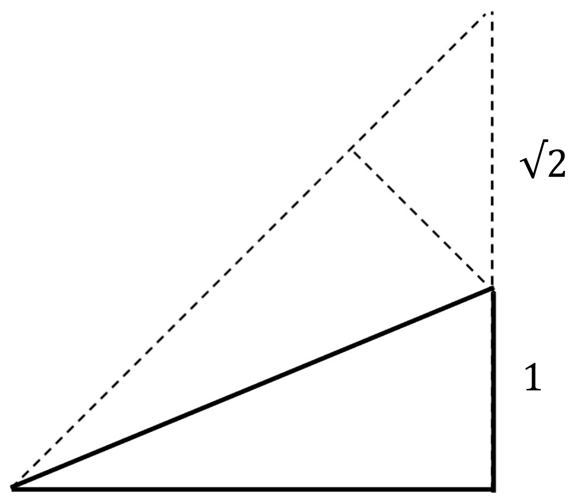
$$A_\iota\left(\frac{\partial}{\partial u}\right) = k \cdot dF\left(\frac{\partial}{\partial u}\right), \quad A_\iota\left(\frac{\partial}{\partial v}\right) = -k \cdot dF\left(\frac{\partial}{\partial v}\right).$$

$\hat{w} = \hat{u} + \sqrt{-1}\hat{v}$: a local complex coordinate of M given by

$$\hat{w} = \exp(\sqrt{-1}\pi/8)w = e^{\sqrt{-1}\theta}w \quad \left(\theta = \frac{\pi}{8}\right).$$

$$\cos\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}(\sqrt{2}+1),$$

$$\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}.$$



Since

$$\frac{\partial}{\partial u} = \frac{\sqrt{2-\sqrt{2}}}{2}\left((\sqrt{2}+1)\frac{\partial}{\partial \hat{u}} + \frac{\partial}{\partial \hat{v}}\right),$$
$$\frac{\partial}{\partial v} = \frac{\sqrt{2-\sqrt{2}}}{2}\left(-\frac{\partial}{\partial \hat{u}} + (\sqrt{2}+1)\frac{\partial}{\partial \hat{v}}\right),$$

we obtain

$$(-4-2\sqrt{2})A_\iota\left(\frac{\partial}{\partial \hat{u}}\right) = -2(\sqrt{2}+1)k\left(\frac{\partial}{\partial \hat{u}} + \frac{\partial}{\partial \hat{v}}\right),$$

which means $\sigma(\hat{T}_1, \hat{T}_1) = \sigma(\hat{T}_1, \hat{T}_2)$ and therefore F is strictly isotropic, by rechoosing the orientation of N if necessary. $\square$

Remark

$F : M \longrightarrow N$: a space-like and conformal immersion with zero mean curvature vector s.t. $\hat{R}(e_1, e_2)\Theta_{F,\pm} = 0$, $d\omega^{\perp} = 0$.

Then we see by the above theorem that F is isotropic if and only if F is strictly isotropic, by rechoosing the orientation of N if necessary.

Remark

Let N be a 4-dimensional Lorentzian space form. Then $\hat{R}(e_1, e_2)\Theta_{F,\pm} = 0$.

In addition, by $Q \equiv 0$, we obtain $d\omega^{\perp} = 0$.

In the next section, we will prove that if F is strictly isotropic, then $Q \equiv 0$.

M: a Lorentz surface,

$F : M \longrightarrow N$: a time-like and conformal immersion,

(e_3, e_4): a local ordered pseudo-orthonormal frame field of TM giving the orientation of M.

Suppose that e_4 is time-like.

(ω^3, ω^4): the dual frame field of (e_3, e_4),

$*$: a linear transformation of T_a^*M defined by $*\omega_3 = \omega_4$, $*\omega_4 = \omega_3$.

Theorem

$F : M \longrightarrow N$: a time-like and conformal immersion with zero mean curvature vector s.t. $\hat{R}(e_3, e_4)\Theta_{F,+} = 0$.

Then the following hold:

(a) *Q is holomorphic;*

(b) *the 2nd fundamental form of F is constructed by solutions of two families of systems of ordinary differential equations defined along integral curves of light-like vector fields $e_3 \pm e_4$ and given by the connection forms $\omega := h(\nabla e_3, e_4)$, $\omega^{\perp} := h(\nabla e_1, e_2)$;*

(c) *if F is strictly isotropic and if F is not totally geodesic on any open set of M, then ω, $\omega^{\perp}$ satisfy $d * \omega = 0$ and $d\omega^{\perp} = 0$ for a suitable (e_3, e_4), and the second fundamental form of F is constructed by a solution of an over-determined system such that the compatibility condition is given by $d * \omega = 0$ and $d\omega^{\perp} = 0$.*

5. Surfaces with zero mean curvature vector in 4-dimensional Lorentzian space forms

N: a 4-dimensional Lorentzian space form,

L_0: the constant sectional curvature of N.

- $L_0 = 0 \implies N = E^4_1 = (\mathbb{R}^4, \langle\ ,\ \rangle_{3,1})$,
$$\langle x, y\rangle_{3,1} = x^1y^1 + x^2y^2 + x^3y^3 - x^4y^4$$
$$(x = (x^1, x^2, x^3, x^4),\ \ y = (y^1, y^2, y^3, y^4)).$$

- $L_0 > 0 \implies N = S^4_1(L_0) = \left\{ x \in E^5_1 \;\middle|\; \langle x, x\rangle_{4,1} = \dfrac{1}{L_0} \right\}.$

- $L_0 < 0 \implies N = H^4_1(L_0) = \left\{ x \in E^5_2 \;\middle|\; \langle x, x\rangle_{3,2} = \dfrac{1}{L_0} \right\}.$

M: a Riemann surface,

$F : M \longrightarrow N$: a space-like and conformal immersion with zero mean curvature vector.

Suppose that F is strictly isotropic.

$w = u + \sqrt{-1}v$: a local complex coordinate of M s.t.

$$K_{F,+}\sigma(T_1, T_1) = \sigma(T_1, T_2)$$

$$\text{for } T_1 := dF\left(\frac{\partial}{\partial u}\right), \ T_2 := dF\left(\frac{\partial}{\partial v}\right).$$

g: the induced metric by F.

We represent g as $g = e^{2\alpha} dw d\overline{w}$.

N_1, N_2: normal vector fields of F s.t.

$$h(N_1, N_1) = e^{2\alpha}, \quad h(N_2, N_2) = -e^{2\alpha}, \quad h(N_1, N_2) = 0.$$

$\Longrightarrow$ $\exists \mu_1$, μ_2, β_1, β_2 s.t.

$$[D_{T_1}F \ D_{T_1}T_1 \ D_{T_1}T_2 \ D_{T_1}N_1 \ D_{T_1}N_2] = [F \ T_1 \ T_2 \ N_1 \ N_2]S,$$
$$[D_{T_2}F \ D_{T_2}T_1 \ D_{T_2}T_2 \ D_{T_2}N_1 \ D_{T_2}N_2] = [F \ T_1 \ T_2 \ N_1 \ N_2]T,$$

where

$$S := \begin{bmatrix} 0 & -L_0e^{2\alpha} & 0 & 0 & 0 \\ 1 & \alpha_u & \alpha_v & -\mu_1 & \mu_2 \\ 0 & -\alpha_v & \alpha_u & -\mu_2 & \mu_1 \\ 0 & \mu_1 & \mu_2 & \alpha_u & \beta_1 \\ 0 & \mu_2 & \mu_1 & \beta_1 & \alpha_u \end{bmatrix}, \quad T := \begin{bmatrix} 0 & 0 & -L_0e^{2\alpha} & 0 & 0 \\ 0 & \alpha_v & -\alpha_u & -\mu_2 & \mu_1 \\ 1 & \alpha_u & \alpha_v & \mu_1 & -\mu_2 \\ 0 & \mu_2 & -\mu_1 & \alpha_v & \beta_2 \\ 0 & \mu_1 & -\mu_2 & \beta_2 & \alpha_v \end{bmatrix}.$$

Since $S_v - T_u = ST - TS$, we obtain

- $\alpha_{uu} + \alpha_{vv} = -L_0 e^{2\alpha}$ (the equation of Gauss),
- $(e^\alpha \mu_p)_u = -e^\alpha \mu_q \beta_1,\ (e^\alpha \mu_p)_v = -e^\alpha \mu_q \beta_2$ for $\{p, q\} = \{1, 2\}$ (the equations of Codazzi),
- $(\beta_1)_v - (\beta_2)_u = 2(\mu_1^2 - \mu_2^2)$ (the equation of Ricci).

Noticing $(e^\alpha \mu_p)_{uv} = (e^\alpha \mu_p)_{vu}$, we obtain $\mu_2 = \pm\mu_1$ and $(\beta_1)_v = (\beta_2)_u$.
From $\mu_2 = \pm\mu_1$, we obtain $Q \equiv 0$.
From $(\beta_1)_v = (\beta_2)_u$, we can find a function ϕ s.t. $\phi_u = \beta_1$, $\phi_v = \beta_2$.
Then by the equations of Codazzi, we can find a constant C s.t. $\mu_1 = Ce^{-\alpha\mp\phi}$.

Theorem (A, 2020)

N: *a 4-dimensional Lorentzian space form,*

L_0: *the constant sectional curvature of* N,

M: *a Riemann surface.*

(a) *For a Hermitian metric* $g = e^{2\alpha} dw d\overline{w}$ *on* M *with constant curvature* L_0 *and a function* ϕ *on* M,

$\exists F$: *a space-like and conformal immersion of a neighborhood of each point of* M *into* N *with zero mean curvature vector satisfying*

- $Q \equiv 0$;
- F *is strictly isotropic, by rechoosing the orientation of* N *if necessary.*

Such an immersion is uniquely determined up to an isometry of N.

(b) $F : M \longrightarrow N$: *a space-like and conformal immersion with zero mean curvature vector.*

If F *is strictly isotropic, then* $Q \equiv 0$.

Remark

N: as in the above theorem,

$F : M \longrightarrow N$: a space-like and conformal immersion with zero mean curvature vector.

$\Longrightarrow$ • $\hat{R}(e_1, e_2)\Omega_{F,\pm} = 0$,

• $Q \equiv 0$ means $d\omega^\perp = 0$.

Therefore F satisfies $Q \equiv 0$ if and only if

F is strictly isotropic, by rechoosing the orientation of N if necessary.

This means that the following are mutually equivalent:

- F is isotropic;
- F is strictly isotropic, by rechoosing the orientation of N if necessary;
- $Q \equiv 0$.

Example

M: a Riemann surface,

$\iota : M \longrightarrow E^3$: a minimal conformal immersion of M into E^3.

$\Longrightarrow$ ι is Willmore and $\tilde{Q} \equiv 0$.

$L^+ := \{x = (x^1, x^2, x^3, x^4, x^5) \in E^5_1 \mid \langle x, x \rangle_{4,1} = 0,\ x^5 > 0\}$.

We consider E^3 to be a subset $L^+ \cap \{x^5 = x^1 + 1\}$ of L^+ and therefore we consider ι to be an L^+-valued function.

γ: the conformal Gauss map of ι,

$\mathrm{Reg}\,(\iota)$: the set of non-umbilical points of ι.

$\Longrightarrow$ • $\gamma|_{\mathrm{Reg}\,(\iota)}$ has zero mean curvature vector,

• the holomorphic quartic differential Q on $\mathrm{Reg}\,(\iota)$ defined by $F = \gamma|_{\mathrm{Reg}\,(\iota)}$ vanishes.

$w = u + \sqrt{-1}v$: a local complex coordinate of $\mathrm{Reg}\,(\iota)$.
We can suppose

- $\partial/\partial u$, $\partial/\partial v$ are in the principal directions of ι,
- $d\gamma\left(\dfrac{\partial}{\partial u}\right) = -\varepsilon d\iota\left(\dfrac{\partial}{\partial u}\right)$, $d\gamma\left(\dfrac{\partial}{\partial v}\right) = \varepsilon d\iota\left(\dfrac{\partial}{\partial v}\right)$,

  where $\varepsilon := \sqrt{-K}$ and K is the Gaussian curvature of ι.

Therefore ι is a light-like normal vector field of $\gamma|_{\mathrm{Reg}\,(\iota)}$ in $S^4_1 = S^4_1(1)$.

A_ι: the shape operator of $\gamma|_{\mathrm{Reg}\,(\iota)}$ w.r.t. ι.

$$\Longrightarrow \quad A_\iota\left(\frac{\partial}{\partial u}\right) = \frac{1}{\varepsilon}d\gamma\left(\frac{\partial}{\partial u}\right), \quad A_\iota\left(\frac{\partial}{\partial v}\right) = -\frac{1}{\varepsilon}d\gamma\left(\frac{\partial}{\partial v}\right).$$

Therefore by $\hat{w} = \exp(\sqrt{-1}\pi/8)w$, we see that $F = \gamma|_{\mathrm{Reg}\,(\iota)}$ is strictly isotropic, by rechoosing the orientation of S^4_1 if necessary.

Remark

$\iota : M \longrightarrow S^3$: a conformal and Willmore immersion,

$\gamma : M \longrightarrow S^4_1$: the conformal Gauss map of ι.

$\Longrightarrow$ • ι is a light-like normal vector field of $\gamma|_{\mathrm{Reg}\,(\iota)}$,

• $\gamma|_{\mathrm{Reg}\,(\iota)}$ has zero mean curvature vector.

Suppose that the holomorphic quartic differential Q on $\mathrm{Reg}\,(\iota)$ defined by $\gamma|_{\mathrm{Reg}\,(\iota)}$ vanishes.

$\Longrightarrow$ A light-like normal vector field ν of $\gamma|_{\mathrm{Reg}\,(\iota)}$ s.t. $\langle \iota, \nu \rangle_{4,1} = -1$ is contained in a constant direction in E^5_1

x_0: a point of S^3 determined by ν

$\Longrightarrow$ The image of $\iota(M) \setminus \{x_0\}$ by the stereographic projection $\mathrm{pr} : S^3 \setminus \{x_0\} \longrightarrow E^3$ from x_0 is a minimal surface in E^3.

Bryant showed that a Willmore sphere in S^3 gives a complete minimal surface in E^3 with finite total curvature s.t. all the ends are embedded and planar.

Based on this result, Kusner constructed complete minimal surfaces Σ_{2k+1} ($k \in \mathbb{N}$) in E^3 given by punctured real projective planes s.t. each Σ_{2k+1} has $2k+1$ planar ends, and inverting them, he gave examples of Willmore projective planes

Referring to these minimal surfaces, Hamada-Kato constructed complete minimal surfaces Σ_{2k+2} ($k \in \mathbb{N}$) in E^3 given by punctured real projective planes s.t. each Σ_{2k+2} has $2k+1$ catenoidal ends and one planar end.

M: a Lorentz surface,

$F : M \longrightarrow N$: a time-like and conformal immersion with zero mean curvature vector.

Suppose that F is strictly isotropic.

$w = u + jv$: a local paracomplex coordinate of M s.t.

$$K_{F,+}\sigma(T_1, T_1) = \sigma(T_1, T_2)$$

$$\text{for } T_1 := dF\left(\frac{\partial}{\partial u}\right),\ \ T_2 := dF\left(\frac{\partial}{\partial v}\right).$$

g: the induced metric by F.

We represent g as $g = e^{2\alpha} dw d\overline{w}$.

N_1, N_2: normal vector fields of F s.t. $h(N_p, N_q) = \delta_{pq} e^{2\alpha}$.

$\Longrightarrow$ $\exists \mu_1, \mu_2, \beta_1, \beta_2$ s.t.

$$[D_{T_1}F \ D_{T_1}T_1 \ D_{T_1}T_2 \ D_{T_1}N_1 \ D_{T_1}N_2] = [F \ T_1 \ T_2 \ N_1 \ N_2]S,$$
$$[D_{T_2}F \ D_{T_2}T_1 \ D_{T_2}T_2 \ D_{T_2}N_1 \ D_{T_2}N_2] = [F \ T_1 \ T_2 \ N_1 \ N_2]T,$$

where

$$S := \begin{bmatrix} 0 & -L_0 e^{2\alpha} & 0 & 0 & 0 \\ 1 & \alpha_u & \alpha_v & -\mu_1 & -\mu_2 \\ 0 & \alpha_v & \alpha_u & -\mu_2 & \mu_1 \\ 0 & \mu_1 & -\mu_2 & \alpha_u & -\beta_1 \\ 0 & \mu_2 & \mu_1 & \beta_1 & \alpha_u \end{bmatrix}, \quad T := \begin{bmatrix} 0 & 0 & L_0 e^{2\alpha} & 0 & 0 \\ 0 & \alpha_v & \alpha_u & \mu_2 & -\mu_1 \\ 1 & \alpha_u & \alpha_v & \mu_1 & \mu_2 \\ 0 & -\mu_2 & \mu_1 & \alpha_v & -\beta_2 \\ 0 & \mu_1 & \mu_2 & \beta_2 & \alpha_v \end{bmatrix}.$$

Since $S_v - T_u = ST - TS$, we obtain

- $\alpha_{uu} - \alpha_{vv} = -L_0 e^{2\alpha}$ (the equation of Gauss),
- $(e^{\alpha}\mu_p)_u = (-1)^{p+1}e^{\alpha}\mu_q\beta_1$, $(e^{\alpha}\mu_p)_v = (-1)^{p+1}e^{\alpha}\mu_q\beta_2$ for $\{p, q\} = \{1, 2\}$ (the equations of Codazzi),
- $(\beta_1)_v - (\beta_2)_u = 2(\mu_1^2 + \mu_2^2)$ (the equation of Ricci).

Noticing $(e^{\alpha}\mu_p)_{uv} = (e^{\alpha}\mu_p)_{vu}$, we obtain $\mu_1 = \mu_2 = 0$ and $(\beta_1)_v = (\beta_2)_u$.

Theorem (A, 2020)

N: *a 4-dimensional Lorentzian space form,*

M: *a Lorentz surface,*

$F : M \longrightarrow N$: *a time-like and conformal immersion with zero mean curvature vector.*

If F is isotropic, then F is totally geodesic.

THANK YOU FOR YOUR ATTENTION!