On the moduli spaces of left-invariant Riemannian metrics on Lie groups

Hiroshi Tamaru (田丸 博士)
Osaka Metropolitan University / OCAMI

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Abstract

Framework

- We define $\mathcal{M}_G$ the moduli space of left-inv. Riem. metrics on a Lie group G.
- Study the (non)existence of distinguished left-inv. metrics on G, in terms of $\mathcal{M}_G$.

Problem (not solved)

- Classify G such that $\dim \mathcal{M}_G = 1$.
- Study left-inv. Ricci soliton metrics on such G, in terms of $\mathcal{M}_G$.

Joint work with

- Masahiro Kawamata
- Yuichiro Taketomi (OMU)

Background - (1/2)

Basic Question

- Which homogeneous spaces G/K admit G-invariant Einstein or Ricci soliton metrics?

Def.

- Riem. mfd (M, g) is **Ricci soliton** if $\exists c \in \mathbb{R}, \exists X \in \mathfrak{X}(M) : \mathrm{Ric}_g = cg + \mathfrak{L}_X g$.

Note

- A situation of homogeneous Einstein/Ricci solitons depends on the signature of c.

Fact (Naber, Petersen-Wylie)

- (M, g) : homogeneous Ricci soliton $(c > 0)$ $\Rightarrow M \cong$ [homog. Einstein with $\mathrm{sc} > 0$] $\times$ [flat].

Fact (cf. Alekseevskii-Kimel'fel'd)

- (M, g) : homogeneous Ricci soliton $(c = 0)$ $\Rightarrow$ flat.

Background - (2/2)

(Generalized) Alekseevskii Conjecture

- (M, g) : homog. Einstein (Ricci soliton), $c < 0$
  $\Rightarrow$ M is a solvmanifold?
  (solvable Lie group with left-invariant metric?)

Fact (Jablonski)

- AC and GAC are equivalent.

Fact (Böhm-Lafuente, 2023)

- AC is true.

Motivating Question

- Which solvable Lie groups admit left-invariant Einstein or Ricci soliton metrics?

Note

- Classification is known only for dim ≤ 6 (Will).
- In higher-dim., the (non-)existence problem is hard to be solved in general.

Framework - (1/3)

Recall

- Study the (non-)existence of left-inv. "nice" metrics on Lie groups.

Difficulty

- For $n := \dim G$ with Lie algebra $\mathfrak{g}$, $\{\text{left-inv. metrics on } G\} \cong GL(n, \mathbb{R})/O(n)$, which is too big...
- Not good to ask: nice metric $\leftrightarrow$ nice point?

Def. (moduli space)

- G : a simply-conn. Lie group with dim. n,
- The **moduli space** $\mathcal{M}_G$ of left-inv. Riem. metrics on G is defined by the orbit space of the action of $\mathbb{R}^\times \mathrm{Aut}(G)$ on $GL(n, \mathbb{R})/O(n)$.

Note

- There is a surjection from $\mathcal{M}_G$ onto $\{\text{left-inv. metrics on } G\}/(\text{isometry and scaling})$.
- It is bijective if G is nilpotent or completely solvable.

Framework - (2/3)

Note

- $\mathcal{M}_G$ is a connected Hausdorff space.
- The points corresponding to principal orbits form a smooth mfd.

Ex (Hashinaga-T. 2017)

- Consider $\mathfrak{g} := \mathrm{span}\{e_1, e_2, e_3\}$ with $a \neq 1$, $[e_1, e_2] = e_2, \ \ [e_1, e_3] = ae_3, \ \ [e_2, e_3] = 0.$
- Then, by a matrix calculation,

$$\mathcal{M}_G \cong \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix} \mid \lambda \geq 0 \right\} \cong [0, +\infty).$$

- On the corresponding G, the metric is Ricci soliton if and only if it corresponds to 0.

Note

- Properties of $\mathcal{M}_G$ are invariants of G.
- Question (Expectation): They reflect the properties of left-inv. metrics on G?

Framework - (3/3)

$g.\langle \ , \ \rangle := \langle g^{-1}(\cdot), g^{-1}(\cdot) \rangle$

Note

- The simplest case is $\dim \mathcal{M}_G = 0$,
  i.e., $\mathcal{M}_G = \{\mathrm{pt}\}$,
  i.e., $\mathbb{R}^\times \mathrm{Aut}(G) \curvearrowright GL(n, \mathbb{R})/O(n)$ transitive.

Thm (Lauret 2003)

- $\dim \mathcal{M}_G = 0$ iff the Lie algebra of G is $\mathbb{R}^n$ (abelian), $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$ (special type), $\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$.

Note

- Above three admit left-inv. Ricci solitons; which can be proved without classification.
- Moreover, it satisfies
  - $\mathbb{R}^n$: flat,
  - $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$: negative constant curvature,
  - $\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$: non-Einstein.
- Hence, left-inv. Einstein metrics could not be characterized in terms of $\mathcal{M}_G$... but how about Ricci solitons?

One-dimensional Case - (1/5)

Conjecture

If $\dim \mathcal{M}_G = 1$, then

- $\mathcal{M}_G \cong \mathbb{R}$ or $[0, +\infty)$,
- $\exists$ left-inv. Ricci soliton iff $\mathcal{M}_G \cong [0, +\infty)$,
- a left-inv. Ricci soliton corresponds to 0.

Note (circumstantial evidence)

- It is true for all examples we know.
- If $\dim \mathcal{M}_G = 1$, then we can show that G is solvable. Therefore left-inv. Ricci soliton is unique (Lauret 2011).

Prop (a partial answer)

- If $\mathcal{M}_G \cong [0, +\infty)$, then 0 corresponds to an isolated orbit.
- A left-inv. metric corresponding to an isolated orbit is Ricci soliton (Taketomi 2022).

One-dimensional Case - (2/5)

$(\mathbb{R}^\times \mathrm{Aut}(G) \cap SL(n,\mathbb{R})) \backslash (SL(n,\mathbb{R})/SO(n))$

Note (ongoing idea)

≅?

- Recall $\mathcal{M}_G := \mathbb{R}^\times \mathrm{Aut}(G) \backslash (GL(n,\mathbb{R})/O(n))$.
- Consider $H \curvearrowright M$: a cohomogeneity one action on an Hadamard mfd.
- If H is connected, then the orbit space $H \backslash M$ is $\mathbb{R}$ or $[0,+\infty)$ (Berndt-Brück 2001).
- In general $\mathbb{R}^\times \mathrm{Aut}(G)$ is not connected ...

- We are trying to classify Lie groups with $\dim \mathcal{M}_G = 1$.
- Many examples are "almost abelian".

Def.

- $\mathfrak{g}$ is **almost abelian** if $\exists$ codimension one abelian ideal. (Namely $\mathfrak{g} \cong \mathbb{R} \ltimes \mathbb{R}^{n-1}$)

One-dimensional Case - (3/5)

Fact

- Each $A \in M(n-1, \mathbb{R})$ gives an almost abelian Lie algebra $\mathfrak{g}_A = \mathrm{span}\{e_1, \ldots, e_n\}$ by

$$\mathrm{ad}_{e_1} = A \quad (\text{on } \mathrm{span}\{e_2, \ldots, e_n\}).$$

- $\mathfrak{g}_A \cong \mathfrak{g}_B$ iff A and B are similar (conjugate up to nonzero scaling).

Ex.

- $\mathbb{R}^n$, $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$, $\mathfrak{h}^3 \oplus \mathbb{R}^{n-3}$ are almost abelian, whose corresponding matrices are

$$O_n, \quad I_n, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{R}^{n-3}.$$

O_{n-3}

Ex.

- Remember $\mathfrak{g} := \mathrm{span}\{e_1, e_2, e_3\}$ with $[e_1, e_2] = e_2, \ [e_1, e_3] = ae_3, \ [e_2, e_3] = 0.$
- This is almost abelian corresponding to

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}.$$

One-dimensional Case - (4/5)

Thm

$\mathfrak{g}_A$ has one-dim. moduli iff A is similar to

- $(\alpha I_{n-2}) \oplus (\beta I_1)$ with $\alpha \neq \beta$;
- $I_{n-3} \oplus \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$;
- $\begin{pmatrix} \gamma & 1 \\ -1 & \gamma \end{pmatrix}$ with $\gamma \geq 0$;
- $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus O_{n-5}$.

Idea of Proof

- $(\Leftarrow)$: direct calculation.
- $(\Rightarrow)$: The action is of cohomogeneity one iff the slice representation is very nice iff A cannot have many eigenvalues...

Conjecture

- $\dim \mathcal{M}_G = 1$ iff its Lie algebra is one of four almost abelian Lie algebras, or $\mathfrak{h}^5 \oplus \mathbb{R}^{n-5}$ (5-dim Heisenberg + abelian).

One-dimensional Case - (5/5)

Recall

For all known examples with one-dim moduli,

- $\mathcal{M}_G \cong \mathbb{R}$ or $[0, +\infty)$,
- $\exists$ left-inv. Ricci soliton iff $\mathcal{M}_G \cong [0, +\infty)$,
- a left-inv. Ricci soliton corresponds to 0.

Problems

Conjecture

If $\mathcal{M}_G$ is one-dimensional, then

- $\mathcal{M}_G \cong \mathbb{R}$ or $[0, +\infty)$,
- $\exists$ left-inv. Ricci soliton iff $\mathcal{M}_G \cong [0, +\infty)$,
- a left-inv. Ricci soliton corresponds to 0.

Note

- The most hard part would be the non-existence in the case of $\mathcal{M}_G \cong \mathbb{R}$;
- Maybe the classification would be easier...?

Other Problems

- Study the case of $\dim \mathcal{M}_G = 2$.
- Study the case that $\mathbb{R}^\times \mathrm{Aut}(G) \curvearrowright GL(n, \mathbb{R})/O(n)$ is "nice" (e.g., polar or hyperpolar).
- Study the pseudo-Riemannian version.

$\mathbb{R}^\times \mathrm{Aut}(G) \curvearrowright G(p+q, \mathbb{R})/O(p, q)$

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Thank you!