Nilpotent Lie algebras obtained by quivers and Ricci solitons

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Abstract

- In the study on homogeneous Ricci solitons, nilpotent Lie groups are important.
- Many studies that construct nilpotent Lie algebras, and examine the existence of left-invariant Ricci solitons.
- Starting from quivers Q, we construct nilpotent Lie algebras $\mathfrak{n}_Q$.
- Thm: If Q is finite and without cycles, then $\mathfrak{n}_Q$ admits a Ricci soliton.

- Joint Work with Fumika Mizoguchi (arXiv:2405.11184) and some others.

Nilmanifolds - (1/3)

Note

- nilmanifold (in this talk)
  := a simply-connected nilpotent Lie group with a left-invariant Riemannian metric

Def.

A Riem mfd (M, g) is **Ricci soliton**
$:\Leftrightarrow \quad \exists c \in \mathbb{R}, \exists X \in \mathfrak{X}(M) : \mathrm{ric} = cg + \mathcal{L}_X g.$

Note

A situation depends on the signature of c:

- $c > 0$: it is **rigid** (flat $\times$ homog Einstein);
- $c = 0$: it must be flat.

Thm (Böhm-Lafente 2023)

(M, g) : homogeneous Ricci soliton with $c < 0$
$\Rightarrow$ (M, g) is a solvmanifold (simply-connected solvable Lie group with a left-inv Riem metric).

Nilmanifolds - (2/3)

Note

We identify:

- (G, g) : a simply-connected Lie group with left-invariant Riem metric;
- $(\mathfrak{g}, \langle, \rangle)$: its Lie algebra with positive definite inner product.

Def.

A Lie algebra $\mathfrak{g}$ with $\mathfrak{g}^0 := \mathfrak{g}$, $\mathfrak{g}^k := [\mathfrak{g}, \mathfrak{g}^{k-1}]$ is

- ***m*-step nilpotent** if $\mathfrak{g}^m = 0$ and $\mathfrak{g}^{m-1} \neq 0$;
- **solvable** if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Ex.

The following are solvable and (m-step) nilpotent:

$$\mathfrak{sl}(m+1)_{\mathbb{R}} \supset \underbrace{\left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\}}_{\text{solvable}} \supset \underbrace{\left\{\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}\right\}}_{\text{nilpotent}}$$

Nilmanifolds - (3/3)

Thm. (Lauret 2011)

- $(\mathfrak{s}, \langle , \rangle)$ (solvable) is Ricci soliton
  $\Rightarrow (\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}], \langle , \rangle|_{\mathfrak{n}\times\mathfrak{n}})$ is Ricci soliton;
- $(\mathfrak{n}, \langle , \rangle)$ (nilpotent) is Ricci soliton
  $\Rightarrow \exists \underbrace{(\mathfrak{s}, \langle , \rangle')\ \text{(solvable)}}_{\text{Ricci soliton}}$: its derived is $(\mathfrak{n}, \langle , \rangle)$.

Note

- The above $(\mathfrak{s}, \langle , \rangle')$ (solvable extension) can be chosen to be Einstein.

Thm. (Lauret 2003)

$(\mathfrak{n}, \langle , \rangle)$ (nilpotent) is Ricci soliton iff

- $\exists c \in \mathbb{R}, \exists D \in \underbrace{\mathrm{Der}}_{\text{derivation}}(\mathfrak{n}) : \mathrm{Ric} = c \cdot \mathrm{id} + D.$

$D[\cdot, \cdot] = [D(\cdot), \cdot] + [\cdot, D(\cdot)]$

Summary

- Nilpotent Ricci soliton $(\mathfrak{n}, \langle , \rangle)$ are important.

Examples - (1/5)

Ex. (Heisenberg)

The standard inner product on $\mathfrak{h}^3$ is Ricci soliton:

$$\mathfrak{h}^3 := \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\} \quad \text{(3-dim Heisenberg).}$$

Ex. 1: rep of Clifford algebra $\to$ nilpotent (H-type)

Note

For two-step nilpotent $(\mathfrak{n}, \langle , \rangle)$,

- $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{z}$ center, $\mathfrak{v} := \mathfrak{z}^\perp$;
- $[,]$ is controlled by $J : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$, where $\langle J_Z(X), Y \rangle = \langle Z, [X, Y] \rangle$.

Def. (Kaplan 1980)

$(\mathfrak{n}, \langle , \rangle)$: two-step nilpotent is of **H-type** if

- $\forall Z \in \mathfrak{z}$, $J_Z^2 = -\langle Z, Z \rangle \cdot \mathrm{id}$.

Examples - (2/5)

Note

H-type condition is equivalent to

- J can be extended to $\widetilde{J} : \mathrm{Cl}(\mathfrak{z}, \langle , \rangle) \to \mathrm{End}(\mathfrak{v})$.

$\mathfrak{z}$ J

Note

- $\mathrm{Cl}_1 \cong \mathbb{C} \curvearrowright \mathbb{C}^n \to (2n+1)$-dim Heisenberg;
- $\mathrm{Cl}_2 \cong \mathbb{H} \curvearrowright \mathbb{H}^n \to$ complex Heisenberg; ...

Thm. (Boggino 1985, Lauret 2003)

- Every H-type Lie algebra is Ricci soliton.

Ex. 2: simple (directed) graph $\to$ two-step nilpotent

Def. (Dani-Mainkar 2005)

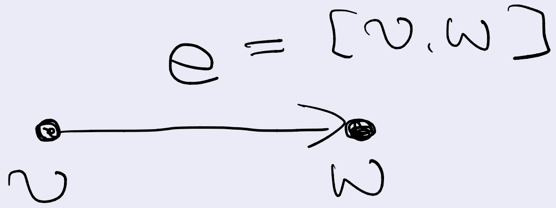
For $G = (V, E)$: simple directed graph ($E \subset V \times V$),

- $\mathfrak{n} := \mathrm{span}(E \cup V)$ is two-step nilpotent by $[v, w] = e$, when e is an edge from v to w.

Examples - (3/5)

Ex.

3-dim Heisenberg $\mathfrak{h}^3$ is obtained by



Thm. (Lauret-Will 2011)

- A graph $G = (V, E)$ is "positive" iff the obtained Lie algebra admits Ricci soliton.

Ex. 3: parabolic subalgebra $\to$ nilpotent

Fact

For a real semisimple Lie algebra $\mathfrak{g}$,

- choosing a subset Φ of simple roots in the restricted root system, one has a parabolic subalgebra $\mathfrak{q}_\Phi$;
- $\mathfrak{q}_\Phi$ has the Langlands decomposition $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ with $\mathfrak{n}_\Phi$ nilpotent.

Examples - (4/5)

- This is a (kind of) generalization of Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$
  ($\mathfrak{q}_\Phi$ minimal parabolic $\Rightarrow \mathfrak{n}_\Phi = \mathfrak{n}$);
- Typical example (for $\mathfrak{sl}(n, \mathbb{R})$) is given by "block decomposition":

$$\mathfrak{sl}(n,\mathbb{R}) \supset \underbrace{\left\{ \left(\begin{array}{c|c|c|c} * & * & * & * \\ \hline 0 & * & * & * \\ \hline 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{array} \right) \right\}}_{\mathfrak{q}_\Phi} \supset \underbrace{\left\{ \left(\begin{array}{c|c|c|c} 0 & * & * & * \\ \hline 0 & 0 & * & * \\ \hline 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{array} \right) \right\}}_{\mathfrak{n}_\Phi}$$

Thm. (T 2011)

- Every $\mathfrak{n}_\Phi$ admits Ricci soliton.

Note

- $\mathfrak{g} = \mathfrak{su}(1, n), \mathfrak{sp}(1, n), \mathfrak{f}_4^{-20} \Rightarrow \mathfrak{n}_\Phi$ is H-type;
- $\mathfrak{n}_\Phi$ can have arbitrary high nilpotency step;
- It was well-known that $\mathfrak{n}$ (Iwasawa nilpotent) admits Ricci soliton.

Examples - (5/5)

Summary

- $(\mathfrak{n}, \langle , \rangle)$ nilpotent Ricci soliton are interesting;
- several examples from different viewpoints;
- far from complete understanding
(complete classification is only for dim ≤ 7);
- not so many examples for higher-step case...

Quivers - (1/3)

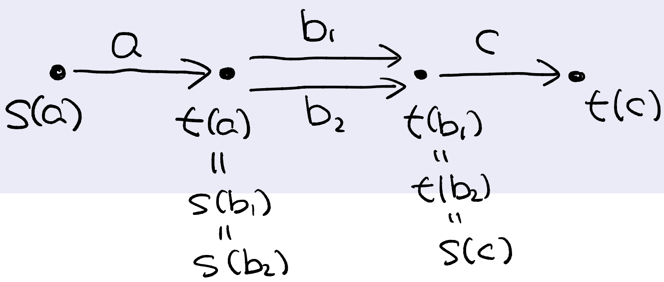
quiver (originally): a container for holding arrows

Def.

$Q = (V, E, s, t)$ is a **quiver** if

- V, E : sets (vertices and edges);
- $s, t : E \to V$: maps (source and target).

Ex.

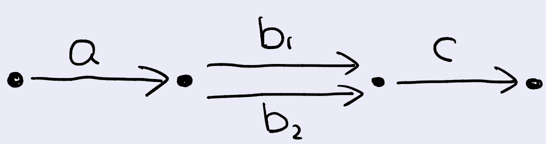


Def.

$\alpha_1 \cdots \alpha_m$ is a **path** of a quiver if

- $\alpha_i \in E$ and $t(\alpha_i) = s(\alpha_{i+1})$ for $\forall i$.

Ex.



path: a, b_1, b_2, c,
ab_1, ab_2, b_1c, b_2c,
ab_1c, ab_2c

Quivers - (2/3)

Def.

For a quiver Q, the **path algebra** is defined by

- Space: $\mathrm{span}(\mathrm{Path}(Q))$,
  where $\mathrm{Path}(Q) := \{$all paths in $Q\}$;
- Product: For $\alpha, \beta \in \mathrm{Path}(Q)$, define

$$\alpha \cdot \beta := \begin{cases} \alpha\beta & (\text{if } t(\alpha) = s(\beta)), \\ 0 & (\text{others}). \end{cases}$$

Def.

For a quiver Q, define the Lie algebra $\mathfrak{n}_Q$ by

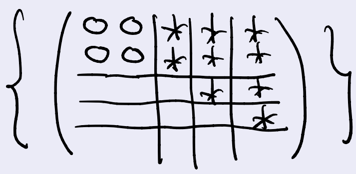
- $\mathfrak{n}_Q := \mathrm{span}(\mathrm{Path}(Q))$, $[\alpha, \beta] := \alpha \cdot \beta - \beta \cdot \alpha$.

Ex.

$$\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \implies \mathfrak{n}_Q = \mathrm{span}\{a, b, ab\}$$

with $[a,b] = a\cdot b - b\cdot a = ab$

$$\bullet \overset{a_1}{\underset{a_2}{\rightrightarrows}} \bullet \xrightarrow{b} \bullet \xrightarrow{c} \bullet \implies \mathfrak{n}_Q \cong$$



$$\bullet \underset{b}{\overset{a}{\rightleftarrows}} \bullet \implies \dim \mathfrak{n}_Q = +\infty$$

Quivers - (3/3)

Def

A path α is a **cycle** if

- $t(\alpha) = s(\alpha)$.

$$\begin{tikzcd} \bullet \arrow[r, shift left, "a"] & \bullet \arrow[l, shift left, "b"] \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \arrow[loop right, "b"] \end{tikzcd}$$

$\alpha = ab \qquad \alpha = b$

Prop.

For a finite quiver Q without cycles,

- $\exists\ m :=$ max of length of paths;
- $\mathfrak{n}_Q$ is an m-step nilpotent Lie algebra.

$(\dim \mathfrak{n}_Q < \infty)$

Note $\{\mathfrak{n}_Q\} \not\subset \{\text{parabolic nilradical}\}$

$\mathfrak{n}_Q \cong \nexists \mathfrak{n}_\Phi$ (parabolic nilradical) for

$$\begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \arrow[r, shift left, "b_1"] \arrow[r, shift right, "b_2"'] & \bullet \arrow[r, "c"] & \bullet \end{tikzcd}$$

path: a, b_1, b_2, c,
ab_1, ab_2, b_1c, b_2c,
ab_1c, ab_2c

Note $\{\mathfrak{n}_Q\} \not\supset \{\text{parabolic nilradical}\}$

$\nexists Q : \mathfrak{n}_Q \cong \mathfrak{h}^5$.

$$= \left\{ \left(\begin{array}{c|cc|c} 0 & * & * & * \\ \hline 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ \hline 0 & 0 & 0 & 0 \end{array} \right) \right\}$$

Result - (1/3)

Thm. (Mizoguchi-T.)

For a finite quiver Q without cycles,

- $\mathfrak{n}_Q$ always admits Ricci soliton.

Note

- This provides many examples of Ricci soliton nilmanifolds with arbitrary high nilpotency.

Idea of Proof

- Construct $\langle , \rangle$ on $\mathfrak{n}_Q$ which is Ricci soliton;
- By induction on the number of steps.

More...

We can construct $\langle , \rangle$ satisfying

(1) $\mathrm{Ric} = -\mathrm{id} + D$ (D derivation);

(2) $\mathrm{Path}(Q)$ is orthogonal;

(3) the norm of a path is invariant by $\mathrm{Aut}(Q)$.

Result - (2/3)

Step 1 ($m = 1$)

Assume $\mathfrak{n}_Q$ is 1-step nilpotent (abelian). Then

- Take $\langle , \rangle$ such that $\mathrm{Path}(Q)$ is orthonormal;
- This satisfies (1)-(3);
- In fact, $\mathrm{Ric} = 0 = -\mathrm{id} + \mathrm{id}$ with id derivation.

Step 2 ($m \to m + 1$): illustration

Assume the claim holds for m-step case; $(m=2)$

Consider Q :

$$\bullet \xrightarrow{a} \bullet \underset{b_2}{\overset{b_1}{\rightrightarrows}} \bullet \xrightarrow{c} \bullet$$

$$\rightsquigarrow \mathrm{Path}(Q) = \{a\} \cup \underbrace{\left\{ \begin{matrix} b_1, b_2, c \\ ab_1, ab_2, b_1c, b_2c, \\ ab_1c, ab_2c \end{matrix} \right\}}_{\mathrm{Path}(Q')}$$

where

$$Q' : \begin{tikzcd} \bullet \arrow[dr, shift left, "ab_1"] \arrow[dr, shift right, "ab_2"'] & & \\ & \bullet \arrow[r, "c"] & \bullet \\ \bullet \arrow[ur, shift left, "b_1"] \arrow[ur, shift right, "b_2"'] & & \end{tikzcd}$$

$\rightsquigarrow$ By assumption Q' admits $\langle , \rangle'$

$\rightsquigarrow$ We can define $\langle a, a \rangle$ as expected.

Recent Studies - (1/3)

Def

For a finite quiver Q without cycles, define

- $\mathrm{Path}_{\geq 0}(Q) := V \cup \mathrm{Path}(Q)$
  (a vertex is regarded as a path of length 0).

Prop.

One can construct a solvable Lie algebra $\mathfrak{s}_Q$ by

- $\mathfrak{s}_Q := \mathrm{span}(\mathrm{Path}_{\geq 0}(Q))$;
- For $\alpha \in \mathrm{Path}(Q)$, $[s(\alpha), \alpha] = [\alpha, t(\alpha)] = \alpha$.

Ex.

$$\underset{v_1}{\bullet} \xrightarrow{a} \underset{v_2}{\bullet} \xrightarrow{b} \underset{v_3}{\bullet} \xrightarrow{c} \underset{v_4}{\bullet} \implies \mathfrak{s}_Q \cong \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\}$$

Prop. (Mizoguchi)

- $\mathfrak{s}_Q$ always admits Ricci soliton;
- Some of them are rigid (flat $\times$ Einstein).

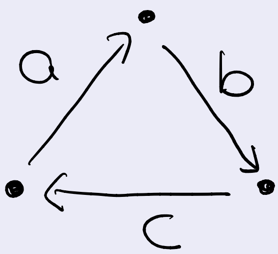
Recent Studies - (2/3)

Def

For a finite quiver Q (possibly with cycles), let

- $\mathrm{APath}(Q) := \{\alpha \in \mathrm{Path}(Q) \mid \alpha \text{ no cycles}\}$.
- $\mathfrak{n}_{AQ}$ can be defined similarly.

Ex.

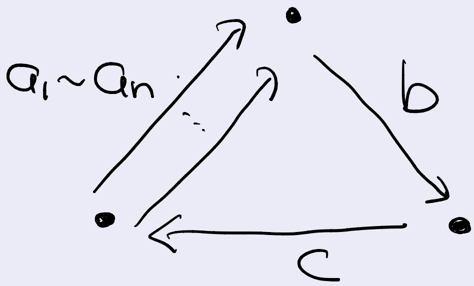


$\Rightarrow$ APath : a , b , c
ab , bc , ca

$\mathfrak{n}_{AQ} \cong \mathbb{R}^3 \oplus \Lambda^2 \mathbb{R}^3$

Ex. (Kitayama) in progress

For the quiver Q:



$\mathfrak{n}_{AQ}$ admits Ricci soliton iff $n \in \{1, 2\}$.

Recent Studies - (3/3)

Note

It will be interesting to study (other) geometric structures on $\mathfrak{n}_Q$.

Announcement

Tomorrow there will be a talk by Mizoguchi about

- pseudo-Riemannian metrics on $\mathfrak{n}_Q$;
- symplectic structures on $\mathfrak{n}_Q$.

Thank you very much!